Doctoral Thesis

Microscopic Theory of the Flux-Flow Hall Effect in Type-II Superconductors

（第二種超伝導体における渦糸フローホール効果の微視的理論）

Hikaru UEKI

Department of Physics, Graduate School of Science, Hokkaido University

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1 Introduction

The Lorentz force on electric currents flowing in magnetic fields has a unique component perpendicular to both the current and field. It generally induces charge redistribution before recovering a steady state to produce a Hall voltage that eventually brings about force balance along the transverse direction. Extensive studies have been performed over the last few decades on this *Hall effect* [1] in metals and semiconductors, especially on the quantum Hall effect in two dimensions [2].

In contrast, we still have little understanding of the phenomena in superconductors. This is because the force on supercurrent itself may easily be overlooked in the presence of the predominant diamagnetic effect by supercurrent obeying Ampère's law. Indeed, the Lorentz force is missing from the Ginzburg–Landau [3] and Eilenberger [4] equations that have been used extensively in the literature [5, 6, 7], and can only be reproduced microscopically as a next-to-leading-order contribution in the expansion of the Gor'kov equations in terms of the quasiclassical parameter $\delta \equiv 1/k_{\mathrm{F}}\xi_0$ [8, 9]. Hence, the physics of the Lorentz force in superconductors remains mostly theoretically unexplored.

This Hall effect in superconductors may be divided into two categories: one in equilibrium with persistent currents [9, 10, 11, 12, 13, 14] and the other in nonequilibrium situations with the motion of vortices and dissipation [6, 15, 16, 17, 18]. The first one is inherent to superconductors and easier to handle but nevertheless has not been paid much attention in the literature. The second one, which is called *the flux-flow Hall effect*, is much more difficult to investigate than the Hall effect in metals and semiconductors and the equilibrium Hall effect in superconductors due to the presences of the spatial inhomogeneity and vortex motion.

Hence, physics in the flux-flow state of type-II superconductors remains a long-standing and unsettled issue theoretically. For example, the early phenomenological theories of Bardeen and Stephen [19] and Nozières and Vinen [20] cannot explain the sign change in the Hall coefficient as a function of temperature/magnetic field observed in the vortex state of a wide variety of materials. A summary of published papers on the sign reversal of the Hall coefficient for different type-II materials is given by Hagen *et al.* in Ref. [17] and Nagaoka *et al.* in Ref. [18]. Also, the nature of the force acting on an isolated moving vortex have been a matter of many discussions. The force has often been called the Lorentz force in some earlier studies. On the other hand, Nozières and Vinen claimed that a moving vortex is driven mainly by the Magnus force and not by the Lorentz force [20].

The purpose of this thesis is to develop a formalism to calculate the transport coefficient of the equilibrium and nonequilibrium Hall effect in superconductors microscopically and also present numerical examples obtained by using it.

First, we focus on the equilibrium Hall effect in type-II superconductors with persistent supercurrents. London included the Lorentz force in his phenomenological equations of superconductivity, and the following relation is given by him: [21, 10]

$$\boldsymbol{E} = \frac{1}{n_{\mathrm{s}}e}\boldsymbol{B} \times \boldsymbol{j}_{\mathrm{s}}, \tag{1.1}$$

where $e < 0$ denotes the electron charge, n_{s} denotes the superfluids density, $\boldsymbol{E}$ is an electric field, $\boldsymbol{B}$ is a magnetic field, and $\boldsymbol{j}_{\mathrm{s}}$ denotes the supercurrent density. Using it, net charge may emerge due to the Hall effect whenever supercurrent flows. In the vortex state of type-II superconductors, the vortex-core may also accumulate charge due to circulating supercurrent. Subsequently, using phenomenological two-fluid equations with the Lorentz force, van Vijfeijken and Staas modified Eq. (1.1) into [22]

$$\boldsymbol{E} = \frac{1}{ne}\boldsymbol{B} \times \boldsymbol{j}_{\mathrm{s}}, \tag{1.2}$$

where n denotes the electron density. On the other hand, early studies on vortex-core charging regard the vortex-core as a normal region and consider its chemical potential difference from outside the core due to the particle–hole asymmetry in the density of states [23, 24, 25]. For example, Khomskii and Freimth presented the following expression for the vortex-core charge: [23]

$$Q = \varepsilon_0 \frac{\Delta^2}{e\varepsilon_{\mathrm{F}}}, \tag{1.3}$$

where ε_0 is the vacuum permittivity, Δ is the energy gap and ε_{F} is the Fermi energy. Although the vortex-core charging itself has been confirmed by microscopic calculations based on the Bogoliubov–de Gennes equations, which is the mean-field theory of superconductivity [26, 27, 28, 29], the origin of the vortex-core charging cannot be clarified. Kumagai *et al.* confirmed the existence of the vortex-core charge experimentally, and observed both the sign and magnitude of the charge by NMR [30].

After the sign change of the Hall conductivity has been observed in some cuprate superconductors [31, 32], intensive investigations have been performed on the flux-flow Hall effect in type-II superconductors both theoretically and experimentally. Despite these efforts, a microscopic understanding of the flux-flow Hall effect is still missing. This may be because the standard Eilenberger equations [4], which have been used extensively to study vortices quantitatively [33, 34, 35, 36] and are now regarded as a basic and reliable tool for investigating inhomogeneous and/or nonequilibrium superconductors microscopically [53, 38, 39, 7], cannot describe the charging and Hall effect.

To overcome this difficulty, several authors have attempted to augment the quasiclassical equations of superconductivity [40, 41, 8]. In 2001, the Lorentz force has been incorporated successfully in a gauge-invariant manner within the real-time Keldysh formalism: [8]

$$\left[\varepsilon\check{\tau}_3 - \check{\Delta} - \check{\sigma}_{\mathrm{imp}}, \check{g}\right]_{\circ} + i\hbar\boldsymbol{v}_{\mathrm{F}}\cdot\boldsymbol{\partial}\check{g} + \frac{i\hbar}{2}\left[e\boldsymbol{v}_{\mathrm{F}}\cdot\boldsymbol{E}\frac{\partial}{\partial\varepsilon} + e\left(\boldsymbol{v}_{\mathrm{F}}\times\boldsymbol{B}\right)\cdot\frac{\partial}{\partial\boldsymbol{p}_{\mathrm{F}}}\right]\{\check{\tau}_3, \check{g}\} = \check{0}, \tag{1.4}$$

where $\check{g} = \check{g}(\varepsilon, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, t)$ is the quasiclassical Green's function in Nambu–Keldysh space, $\check{\Delta} = \check{\Delta}(\boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, t)$ is the pair potential in Nambu–Keldysh space, $\check{\sigma}_{\mathrm{imp}} = \check{\sigma}_{\mathrm{imp}}(\varepsilon, \boldsymbol{r}, t)$ is the impurity self-energy in the self-consistent Born approximation in Nambu–Keldysh space, ε is the excitation energy, $\boldsymbol{\partial}$ is the gauge invariant space derivative, $\boldsymbol{v}_{\mathrm{F}}$ is the Fermi velocity, and $\boldsymbol{p}_{\mathrm{F}}$ is the Fermi momentum. Matrix $\check{\tau}_3$ denotes

$$\check{\tau}_3 = \begin{bmatrix} \hat{\tau}_3 & \hat{0} \\ \hat{0} & \hat{\tau}_3 \end{bmatrix}, \quad \hat{\tau}_3 = \begin{bmatrix} \underline{\sigma}_0 & 0 \\ 0 & -\underline{\sigma}_0 \end{bmatrix}, \tag{1.5}$$

where $\underline{\sigma}_0$ is the 2×2 unit matrix. Notations $[a, b]_\circ$ and $\{a, b\}$ in Eq. (1.4) are given by $[a, b]_\circ \equiv a \circ b - b \circ a$ and $\{a, b\} \equiv ab + ba$ with

$$a \circ b \equiv \exp\left[\frac{i\hbar}{2}\left(\frac{\partial}{\partial \varepsilon}\partial_{t'} - \partial_t \frac{\partial}{\partial \varepsilon'}\right)\right] a(\varepsilon, t) b(\varepsilon', t')\Big|_{\varepsilon'=\varepsilon, t'=t}. \tag{1.6}$$

Here, ∂_t is the gauge invariant time derivative. Equation (1.4) reduces to the standard Eilenberger equations by omitting the last two terms with $\boldsymbol{E}$ and $\boldsymbol{B}$, and the quasiclassical Boltzmann equation in static electromagnetic fields by taking normal state limit of $\check{\Delta} \to \check{0}$. The augmented quasiclassical equations in the Keldysh formalism have been used to study charging in the Meissner state with Fermi surface and gap anisotropies [10], and the flux-flow Hall conductivity for the s-wave pairing on an isotropic Fermi surface [16]. However, the temperature/magnetic field dependence of the Hall conductivity have not been calculated in Ref. [16]

Also, we derived augmented quasiclassical equations of superconductivity with the Lorentz force in the Matsubara formalism so as to calculate the vortex-core charging as a function of temperature/magnetic field for the s-wave on an isotropic Fermi surface [43] and d-pairing on an anisotropic Fermi surface [9, 44]. The augmented quasiclassical equations of superconductivity with the Lorentz force in the Matsubara formalism are given by [9]

$$\left[i\varepsilon_n \hat{\tau}_3 - \hat{\Delta}\hat{\tau}_3 - \hat{\sigma}_{\mathrm{imp}}\hat{\tau}_3, \hat{g}\right] + i\hbar \boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\partial}\hat{g} + \frac{i\hbar}{2} e(\boldsymbol{v}_{\mathrm{F}} \times \boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \{\hat{\tau}_3, \hat{g}\} = \hat{0}, \tag{1.7}$$

where $\hat{g} = \hat{g}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})$ is the quasiclassical Green's function in the Nambu space, $\hat{\Delta} = \hat{\Delta}(\boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})$ is the pair potential in the Nambu space, $\hat{\sigma}_{\mathrm{imp}} = \hat{\sigma}_{\mathrm{imp}}(\varepsilon_n, \boldsymbol{r})$ is the impurity self-energy in the self-consistent Born approximation in the Nambu space and $\varepsilon_n = (2n + 1)\pi k_{\mathrm{B}} T$ is the fermion Matsubara energy ($n = 0, \pm 1, \pm 2, \dots$) with k_{B} and T denoting the Boltzmann constant and temprature. Notation $[a, b]$ is given by $[a, b] \equiv ab - ba$. Then, the expression for the charge density in the Matsubara formalism needs to be modified into

$$\rho = -\frac{i}{2}\pi k_{\mathrm{B}} T e N(0) \sum_{n=-\infty}^{\infty} \mathrm{Tr}\langle \hat{g} \rangle_{\mathrm{F}} - 2e^2 N(0)\Phi, \tag{1.8}$$

where $N(0)$ is the normal-state density of states per spin and unit volume at the Fermi level and Φ is the scalar potential. This expression is the same as that in Refs. [39, 45, 46]. It is still desirable when studying the charging to transform the equations into the Matsubara formalism, in which equilibrium properties and linear responses can be calculated much more easily.

We study the vortex-core charging due to the Lorentz force based on the augmented quasiclassical equations of the superconductivity with the Lorentz force. We also study the flux-flow Hall effect in a superconductor with an isolated vortex based on the augmented quasiclassical equations of the superconductivity with the Lorentz force. In particular, we calculate the longitudinal and Hall electric field induced by an isolated moving vortex by transforming the energy variable of the augmented quasiclassical equations in the Keldysh formalism into the Matsubara energy on the imaginary axis. It is shown that linear responses can be calculated much more easily compared to the approach based on the augmented quasiclassical equations in the Keldysh formalism.

This thesis is organized as follows. In Sect. 2, we present a fully self-consistent calculation of the longitudinal and Hall electric field induced by a motion of a vortex within the Matsubara formalism, in contrast to the conventional approach based on the augmented quasiclassical equations of superconductivity in the Keldysh formalism. In Sect. 3, we present numerical results for vortex-core charging in s-wave and d-wave superconductors. In Sect. 4, we present numerical results for flux-flow Hall effect in s-wave superconductors with an isolated vortex. In Sect. 5, we provide a brief summary and conclusion.

2 Formulation

2.1 Notations

For simplicity, we first restrict ourselves to the spin-singlet pairing without spin paramagnetism and approximate the normal-state density of states per spin and unit volume as $N(\varepsilon) \approx N(0)$. Functions $\check{g}$, $\check{\Delta}$ and $\check{\sigma}_{\rm imp}$ in Eq. (1.7) can be written as

$$\check{g} = \begin{bmatrix} \hat{g}^{\rm R} & \hat{g}^{\rm K} \\ \hat{0} & \hat{g}^{\rm A} \end{bmatrix}, \quad \check{\Delta} = \begin{bmatrix} \hat{\Delta} & \hat{0} \\ \hat{0} & \hat{\Delta} \end{bmatrix}, \quad \check{\sigma}_{\rm imp} = -\frac{i\hbar}{2\tau}\langle \check{g} \rangle_{\rm F}, \tag{2.1}$$

where τ is the relaxation time and $\langle \cdots \rangle_{\rm F}$ denotes the Fermi surface average normalized as $\langle 1 \rangle_{\rm F} = 1$. The 2×2 retarded and Keldysh Green's functions, 2×2 pair potential can be written as

$$\hat{g}^{\rm R,A} = \begin{bmatrix} g^{\rm R,A} & -if^{\rm R,A} \\ i\bar{f}^{\rm R,A} & -\bar{g}^{\rm R,A} \end{bmatrix}, \quad \hat{g}^{\rm K} = \begin{bmatrix} g^{\rm K} & -if^{\rm K} \\ -i\bar{f}^{\rm K} & \bar{g}^{\rm K} \end{bmatrix}, \quad \hat{\Delta} = \begin{bmatrix} 0 & -\Delta \\ \bar{\Delta} & 0 \end{bmatrix}, \tag{2.2}$$

where the barred functions are defined generally by

$$\bar{g}^{\rm R}(\varepsilon, \boldsymbol{p}_{\rm F}, \boldsymbol{r}, t) \equiv g^{\rm R*}(-\varepsilon, -\boldsymbol{p}_{\rm F}, \boldsymbol{r}, t). \tag{2.3}$$

Matrix $\check{\tau}_3$ denotes

$$\check{\tau}_3 = \begin{bmatrix} \hat{\tau}_3 & \hat{0} \\ \hat{0} & \hat{\tau}_3 \end{bmatrix}, \quad \hat{\tau}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.4}$$

We use the gauge $\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t$ and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ with $\Phi = 0$, where $\boldsymbol{A}$ and Φ are the vector and scalar potentials. Notations $[a, b]_\circ$ and $\{a, b\}$ in Eq. (1.7) are given by $[a, b]_\circ \equiv a \circ b - b \circ a$ and $\{a, b\} \equiv ab + ba$ with

$$a \circ b \equiv \exp\left[\frac{i\hbar}{2}\left(\frac{\partial}{\partial \varepsilon}\frac{\partial}{\partial t'} - \frac{\partial}{\partial t}\frac{\partial}{\partial \varepsilon'}\right)\right] a(\varepsilon, t) b(\varepsilon', t')\bigg|_{\varepsilon'=\varepsilon, t'=t} \tag{2.5}$$

The gauge invariant derivative $\boldsymbol{\partial}$ is defined by

$$\boldsymbol{\partial} \equiv \begin{cases} \boldsymbol{\nabla} & \text{on } g^{\rm R,A,K}, \bar{g}^{\rm R,A,K} \\ \boldsymbol{\nabla} - i\dfrac{2e\boldsymbol{A}}{\hbar} & \text{on } f^{\rm R,A,K}, \Delta \\ \boldsymbol{\nabla} + i\dfrac{2e\boldsymbol{A}}{\hbar} & \text{on } \bar{f}^{\rm R,A,K}, \bar{\Delta} \end{cases}. \tag{2.6}$$

The electromagnetic fields obey Gauss' and Ampère's laws:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \quad \rho = -\frac{eN(0)}{2}\int_{-\infty}^{\infty} \langle g^{\rm K} \rangle_{\rm F} d\varepsilon, \tag{2.7}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j}, \quad \boldsymbol{j} = -\frac{eN(0)}{2}\int_{-\infty}^{\infty} \langle \boldsymbol{v}_{\rm F} g^{\rm K} \rangle_{\rm F} d\varepsilon, \tag{2.8}$$

where ρ and $\boldsymbol{j}$ are the charge and current densities, respectively. Using the basis function on the Fermi surface $\phi = \phi(\boldsymbol{p}_{\rm F})$ normalized as $\langle|\phi|^2\rangle_{\rm F} = 1$, we rewrite the pair potential as $\Delta(\boldsymbol{p}_{\rm F}, \boldsymbol{r}, t) \to \Delta(\boldsymbol{r}, t)\phi(\boldsymbol{p}_{\rm F})$; the equation of $\Delta = \Delta(\boldsymbol{r}, t)$ is given by

$$\Delta = \frac{\mathrm{g}_0}{4i}\int_{-\infty}^{\infty}\langle f^{\rm K}\phi^*\rangle_{\rm F} d\varepsilon \tag{2.9}$$

where the coupling constant g_0 is expressible alternatively as

$$\frac{1}{\mathrm{g}_0} = \ln\frac{T}{T_{\rm c}} + \int_{-\infty}^{\infty}\frac{1}{2\varepsilon}\tanh\frac{\varepsilon}{2k_{\rm B}T}d\varepsilon \tag{2.10}$$

Now, we use the following relations:

$$i\hbar\boldsymbol{v}_{\rm F}\cdot\boldsymbol{\partial}\check{g} = i\hbar\boldsymbol{v}_{\rm F}\cdot\boldsymbol{\nabla}\check{g} + [e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{A}\check{\tau}_3, \check{g}]\,, \tag{2.11a}$$

$$[e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{A}\check{\tau}_3, \check{g}] + \frac{i\hbar}{2}e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{E}\frac{\partial}{\partial\varepsilon}\{\check{\tau}_3, \check{g}\} = [e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{A}\check{\tau}_3, \check{g}]_\circ\,, \tag{2.11b}$$

to rewrite Eq. (1.7) as

$$\left[(\varepsilon + e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{A})\check{\tau}_3 - \check{\Delta} - \check{\sigma}_{\rm imp}, \check{g}\right]_\circ + i\hbar\boldsymbol{v}_{\rm F}\cdot\boldsymbol{\nabla}\check{g} + \frac{i\hbar}{2}e\left(\boldsymbol{v}_{\rm F}\times\boldsymbol{B}\right)\cdot\frac{\partial}{\partial\boldsymbol{p}_{\rm F}}\{\check{\tau}_3, \check{g}\} = \check{0}. \tag{2.12}$$

2.2 Double expansions in the external field and quasiclassical parameter

We study the dissipative Hall effect in type-II superconductors caused by the motion of an isolated vortex by considering the linear response $\check{g} = \check{g}^{\rm eq} + \delta\check{g}$ to a spatially uniform but time-dependent perturbation $\delta\boldsymbol{A}^{\rm ex}\mathrm{e}^{-i\omega t} = \delta\boldsymbol{E}^{\rm ex}\mathrm{e}^{-i\omega t}/i\omega$ with frequency ω [42]. The limit $\omega \to 0$ will be taken eventually. It follows from Eq. (2.12) that the equilibrium functions $\check{g}^{\rm eq} = \check{g}^{\rm eq}(\varepsilon, \boldsymbol{p}_{\rm F}, \boldsymbol{r})$ obey

$$\left[(\varepsilon + e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{A}^{\rm eq})\check{\tau}_3 - \check{\Delta}^{\rm eq} - \check{\sigma}^{\rm eq}_{\rm imp}, \check{g}^{\rm eq}\right]_\circ + i\hbar\boldsymbol{v}_{\rm F}\cdot\boldsymbol{\nabla}\check{g}^{\rm eq} + \frac{i\hbar}{2}e\left(\boldsymbol{v}_{\rm F}\times\boldsymbol{B}^{\rm eq}\right)\cdot\frac{\partial}{\partial\boldsymbol{p}_{\rm F}}\{\check{\tau}_3, \check{g}^{\rm eq}\} = \check{0}. \tag{2.13}$$

and equation for the first-order response $\delta\check{g} = \delta\check{g}(\varepsilon, \boldsymbol{p}_{\rm F}, \boldsymbol{r}, t)$ is given by

$$\begin{aligned}&\left[(\varepsilon + e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{A}^{\rm eq})\check{\tau}_3 - \check{\Delta}^{\rm eq} - \check{\sigma}^{\rm eq}_{\rm imp}, \delta\check{g}\right]_\circ + \left[e\boldsymbol{v}_{\rm F}\cdot\delta\boldsymbol{A}\check{\tau}_3 - \delta\check{\Delta} - \delta\check{\sigma}_{\rm imp}, \check{g}^{\rm eq}\right]_\circ + i\hbar\boldsymbol{v}_{\rm F}\cdot\boldsymbol{\nabla}\delta\check{g}\\ &+ \frac{i\hbar}{2}e\left(\boldsymbol{v}_{\rm F}\times\boldsymbol{B}^{\rm eq}\right)\cdot\frac{\partial}{\partial\boldsymbol{p}_{\rm F}}\{\check{\tau}_3, \delta\check{g}\} + \frac{i\hbar}{2}e\left(\boldsymbol{v}_{\rm F}\times\delta\boldsymbol{B}\right)\cdot\frac{\partial}{\partial\boldsymbol{p}_{\rm F}}\{\check{\tau}_3, \check{g}^{\rm eq}\} = \check{0}.\end{aligned} \tag{2.14}$$

Hereafter, we remove superscript eq from these equations.

As shown in Refs. [10, 9], the Hall effect in superconductors emerges as the first-order corrections in the quasiclassical parameter $\delta \equiv \hbar/\langle p_{\rm F}\rangle_{\rm F}\xi_0$ ($\ll 1$), where ξ_0 is the coherence length defined in terms of the zero-temperature energy gap $\langle\Delta_0\rangle_{\rm F}$ at $B = 0$ by $\xi_0 \equiv \hbar\langle v_{\rm F}\rangle_{\rm F}/\langle\Delta_0\rangle_{\rm F}$. Noting this fact, we expand Green's functions also in $\delta \equiv \hbar/\langle p_{\rm F}\rangle_{\rm F}\xi_0$

as $\check{g} = \check{g}_0 + \check{g}_1 + \cdots$ and $\delta\check{g} = \delta\check{g}_0 + \delta\check{g}_1 + \cdots$, where $\check{g}_0$, $\check{g}_1$, $\delta\check{g}_0$ and $\delta\check{g}_1$ are defined by

$$\check{g}_0 = \begin{bmatrix} \hat{g}_0^{\rm R} & \hat{g}_0^{\rm K} \\ \hat{0} & \hat{g}_0^{\rm A} \end{bmatrix}, \quad \hat{g}_0^{\rm R,A} = \begin{bmatrix} g_0^{\rm R,A} & -if_0^{\rm R,A} \\ i\bar{f}_0^{\rm R,A} & -\bar{g}_0^{\rm R,A} \end{bmatrix}, \quad \hat{g}_0^{\rm K} = \begin{bmatrix} g_0^{\rm K} & -if_0^{\rm K} \\ -i\bar{f}_0^{\rm K} & \bar{g}_0^{\rm K} \end{bmatrix}, \tag{2.15a}$$

$$\check{g}_1 = \begin{bmatrix} \hat{g}_1^{\rm R} & \hat{g}_1^{\rm K} \\ \hat{0} & \hat{g}_1^{\rm A} \end{bmatrix}, \quad \hat{g}_1^{\rm R,A} = \begin{bmatrix} g_1^{\rm R,A} & -if_1^{\rm R,A} \\ i\bar{f}_1^{\rm R,A} & -\bar{g}_1^{\rm R,A} \end{bmatrix}, \quad \hat{g}_1^{\rm K} = \begin{bmatrix} g_1^{\rm K} & -if_1^{\rm K} \\ -i\bar{f}_1^{\rm K} & \bar{g}_1^{\rm K} \end{bmatrix}, \tag{2.15b}$$

$$\delta\check{g}_0 = \begin{bmatrix} \delta\hat{g}_0^{\rm R} & \delta\hat{g}_0^{\rm K} \\ \hat{0} & \delta\hat{g}_0^{\rm A} \end{bmatrix}, \quad \delta\hat{g}_0^{\rm R,A} = \begin{bmatrix} \delta g_0^{\rm R,A} & -i\delta f_0^{\rm R,A} \\ i\delta\bar{f}_0^{\rm R,A} & -\delta\bar{g}_0^{\rm R,A} \end{bmatrix}, \quad \delta\hat{g}_0^{\rm K} = \begin{bmatrix} \delta g_0^{\rm K} & -i\delta f_0^{\rm K} \\ -i\delta\bar{f}_0^{\rm K} & \delta\bar{g}_0^{\rm K} \end{bmatrix}, \tag{2.15c}$$

$$\delta\check{g}_1 = \begin{bmatrix} \delta\hat{g}_1^{\rm R} & \delta\hat{g}_1^{\rm K} \\ \hat{0} & \delta\hat{g}_1^{\rm A} \end{bmatrix}, \quad \delta\hat{g}_1^{\rm R,A} = \begin{bmatrix} \delta g_1^{\rm R,A} & -i\delta f_1^{\rm R,A} \\ i\delta\bar{f}_1^{\rm R,A} & -\delta\bar{g}_1^{\rm R,A} \end{bmatrix}, \quad \delta\hat{g}_1^{\rm K} = \begin{bmatrix} \delta g_1^{\rm K} & -i\delta f_1^{\rm K} \\ -i\delta\bar{f}_1^{\rm K} & \delta\bar{g}_1^{\rm K} \end{bmatrix}. \tag{2.15d}$$

We also expand the electric field and charge density formally in the quasiclassical parameter δ as $\rho = \rho_0 + \rho_1 + \cdots$, $\delta\rho = \delta\rho_0 + \delta\rho_1 + \cdots$, $\boldsymbol{E} = \boldsymbol{E}_0 + \boldsymbol{E}_1 + \cdots$ and $\delta\boldsymbol{E} = \delta\boldsymbol{E}_0 + \delta\boldsymbol{E}_1 + \cdots$ with $\rho_0 = 0$ and $\boldsymbol{E}_0 = \boldsymbol{0}$ [10, 9]. Equation (2.14) for the equilibrium functions can be sorted according to the order in δ into the two equations:

$$\left[(\varepsilon + e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{A})\check{\tau}_3 - \check{\Delta} - \check{\sigma}_{\rm imp}, \check{g}_0\right] + i\hbar\boldsymbol{v}_{\rm F}\cdot\boldsymbol{\nabla}\check{g}_0 = \check{0}, \tag{2.16a}$$

$$\begin{aligned}&\left[(\varepsilon + e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{A})\check{\tau}_3 - \check{\Delta} - \check{\sigma}_{\rm imp}, \check{g}_1\right] + i\hbar\boldsymbol{v}_{\rm F}\cdot\boldsymbol{\nabla}\check{g}_1 \\ &\quad + \frac{i\hbar}{2}\left[e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{E}_1\frac{\partial}{\partial\varepsilon} + e\left(\boldsymbol{v}_{\rm F}\times\boldsymbol{B}\right)\cdot\frac{\partial}{\partial\boldsymbol{p}_{\rm F}}\right]\{\check{\tau}_3, \check{g}_0\} = \check{0}.\end{aligned} \tag{2.16b}$$

Similarly, Eq. (2.14) for the first-order response can be classified in terms of δ as

$$\left[(\varepsilon + e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{A})\check{\tau}_3 - \check{\Delta} - \check{\sigma}_{\rm imp}, \delta\check{g}_0\right]_\circ + \left[e\boldsymbol{v}_{\rm F}\cdot\delta\boldsymbol{A}\check{\tau}_3 - \delta\check{\Delta} - \delta\check{\sigma}_{\rm imp}, \check{g}_0\right]_\circ + i\hbar\boldsymbol{v}_{\rm F}\cdot\boldsymbol{\nabla}\delta\check{g}_0 = \check{0}, \tag{2.17a}$$

$$\begin{aligned}&\left[(\varepsilon + e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{A})\check{\tau}_3 - \check{\Delta} - \check{\sigma}_{\rm imp}, \delta\check{g}_1\right]_\circ + \left[e\boldsymbol{v}_{\rm F}\cdot\delta\boldsymbol{A}\check{\tau}_3 - \delta\check{\Delta} - \delta\check{\sigma}_{\rm imp}, \check{g}_1\right]_\circ \\ &\quad + i\hbar\boldsymbol{v}_{\rm F}\cdot\boldsymbol{\nabla}\delta\check{g}_1 + \frac{i\hbar}{2}\left[e\boldsymbol{v}_{\rm F}\cdot\boldsymbol{E}_1\frac{\partial}{\partial\varepsilon} + e\left(\boldsymbol{v}_{\rm F}\times\boldsymbol{B}\right)\cdot\frac{\partial}{\partial\boldsymbol{p}_{\rm F}}\right]\{\check{\tau}_3, \delta\check{g}_0\} \\ &\quad + \frac{i\hbar}{2}\left[e\boldsymbol{v}_{\rm F}\cdot\delta\boldsymbol{E}_1\frac{\partial}{\partial\varepsilon} + e\left(\boldsymbol{v}_{\rm F}\times\delta\boldsymbol{B}\right)\cdot\frac{\partial}{\partial\boldsymbol{p}_{\rm F}}\right]\{\check{\tau}_3, \check{g}_0\} = \check{0},\end{aligned} \tag{2.17b}$$

where $[a, b]$ is defined as $[a, b] \equiv ab - ba$. Note that $\delta\boldsymbol{E}_0$ in these equations is given by $\delta\boldsymbol{E}_0 = -\partial\delta\boldsymbol{A}/\partial t$. The normalization conditions for Eq. (2.16a) and Eq. (2.17a) are $\check{g}_0^2 = \check{1}$ and $\check{g}_0 \circ \delta\check{g}_0 + \delta\check{g}_0 \circ \check{g}_0 = \check{0}$ [45].

2.3 Eschrig's transport equations without Hall terms

We first consider Eqs. (2.16a) and (2.17a) that are zeroth order in the quasisclassical parameter δ, i.e., the equations without the Hall terms. To obtain their numerical solutions by removing unphysical solutions that explode exponentially as we proceed with the numerical integration, we transform these equations by following the method suggested by Eschrig *et al.* [45]. We use the notation of Ref. [45] with a replacement of

$g_0^{\mathrm{R,A,K}} \to -i\pi g_0^{\mathrm{R,A,K}}$, $\delta g_0^{\mathrm{R,A,K}} \to -i\pi\delta g_0^{\mathrm{R,A,K}}$, $f_0^{\mathrm{R,A,K}} \to -\pi f_0^{\mathrm{R,A,K}}$, $\delta f_0^{\mathrm{R,A,K}} \to -\pi\delta f_0^{\mathrm{R,A,K}}$, $\Delta \to -\Delta$, $\delta\Delta \to -\delta\Delta$, $\gamma^{\mathrm{R,A}} \to -i\gamma^{\mathrm{R,A}}$ and $\delta\gamma^{\mathrm{R,A}} \to -i\delta\gamma^{\mathrm{R,A}}$.

Equation (2.16a) is the main part of Eilenberger equations. The solution $\hat{g}_0^{\mathrm{K}}$ satisfies $\hat{g}_0^{\mathrm{K}} = (\hat{g}^{\mathrm{R}} - \hat{g}^{\mathrm{A}})\tanh(\varepsilon/2k_{\mathrm{B}}T)$. Let us express $g_0^{\mathrm{R,A}} = g_0^{\mathrm{R,A}}(\varepsilon, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})$ and $f_0^{\mathrm{R,A}} = f_0^{\mathrm{R,A}}(\varepsilon, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})$ alternatively as

$$g_0^{\mathrm{R}} = \frac{1-\gamma^{\mathrm{R}}\bar{\gamma}^{\mathrm{R}}}{1+\gamma^{\mathrm{R}}\bar{\gamma}^{\mathrm{R}}}, \tag{2.18a}$$

$$g_0^{\mathrm{A}} = -\frac{1-\gamma^{\mathrm{A}}\bar{\gamma}^{\mathrm{A}}}{1+\gamma^{\mathrm{A}}\bar{\gamma}^{\mathrm{A}}}, \tag{2.18b}$$

$$f_0^{\mathrm{R}} = \frac{2\gamma^{\mathrm{R}}}{1+\gamma^{\mathrm{R}}\bar{\gamma}^{\mathrm{R}}}, \tag{2.18c}$$

$$f_0^{\mathrm{A}} = -\frac{2\gamma^{\mathrm{A}}}{1+\gamma^{\mathrm{A}}\bar{\gamma}^{\mathrm{A}}}, \tag{2.18d}$$

with $\gamma^{\mathrm{R,A}} = \gamma^{\mathrm{R,A}}(\varepsilon, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})$. Equation for $\gamma^{\mathrm{R,A}}$ is given as

$$\begin{aligned}\hbar\boldsymbol{v}_{\mathrm{F}} \cdot \left(\boldsymbol{\nabla} - i\frac{2e\boldsymbol{A}}{\hbar}\right)\gamma^{\mathrm{R,A}} + 2\left[-i\varepsilon + \frac{\hbar}{4\tau}(\langle g_0^{\mathrm{R,A}}\rangle_{\mathrm{F}} + \langle \bar{g}_0^{\mathrm{R,A}}\rangle)\right]\gamma^{\mathrm{R,A}} \\ + \left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_0^{\mathrm{R,A}}\rangle_{\mathrm{F}}\right)\gamma^{\mathrm{R,A}2} - \left(\Delta\phi + \frac{\hbar}{2\tau}\langle f_0^{\mathrm{R,A}}\rangle_{\mathrm{F}}\right) = 0,\end{aligned} \tag{2.19}$$

where $g_0^{\mathrm{A}} = -g_0^{\mathrm{R}*}$, $f_0^{\mathrm{A}} = \bar{f}_0^{\mathrm{R}*}$, $\gamma^{\mathrm{A}} = -\bar{\gamma}^{\mathrm{R}*}$ are obtained from the symmetry relation $\hat{g}_0^{\mathrm{A}} = -\hat{\tau}_3\hat{g}_0^{\mathrm{R}\dagger}\hat{\tau}_3$.

Next, we consider the first-order equations $\delta\boldsymbol{A}$ for the retarded and advanced Green's functions $\delta g_0^{\mathrm{R,A}}$ and $\delta f_0^{\mathrm{R,A}}$, which are relevant to the change in the density of states. We express $\delta g_0^{\mathrm{R,A}} = \delta g_0^{\mathrm{R,A}}(\varepsilon, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, t)$ and $\delta f_0^{\mathrm{R,A}} = \delta f_0^{\mathrm{R,A}}(\varepsilon, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, t)$ alternatively as

$$\delta g_0^{\mathrm{R}} = 2\frac{1}{1+\gamma^{\mathrm{R}}\bar{\gamma}^{\mathrm{R}}} \circ (-\delta\gamma^{\mathrm{R}} \circ \bar{\gamma}^{\mathrm{R}} - \gamma^{\mathrm{R}} \circ \delta\bar{\gamma}^{\mathrm{R}}) \circ \frac{1}{1+\gamma^{\mathrm{R}}\bar{\gamma}^{\mathrm{R}}}, \tag{2.20a}$$

$$\delta g_0^{\mathrm{A}} = -2\frac{1}{1+\gamma^{\mathrm{A}}\bar{\gamma}^{\mathrm{A}}} \circ (-\delta\gamma^{\mathrm{A}} \circ \bar{\gamma}^{\mathrm{A}} - \gamma^{\mathrm{A}} \circ \delta\bar{\gamma}^{\mathrm{A}}) \circ \frac{1}{1+\gamma^{\mathrm{A}}\bar{\gamma}^{\mathrm{A}}}, \tag{2.20b}$$

$$\delta f_0^{\mathrm{R}} = 2\frac{1}{1+\gamma^{\mathrm{R}}\bar{\gamma}^{\mathrm{R}}} \circ (\delta\gamma^{\mathrm{R}} - \gamma^{\mathrm{R}} \circ \delta\bar{\gamma}^{\mathrm{R}} \circ \gamma^{\mathrm{R}}) \circ \frac{1}{1+\gamma^{\mathrm{R}}\bar{\gamma}^{\mathrm{R}}}, \tag{2.20c}$$

$$\delta f_0^{\mathrm{A}} = -2\frac{1}{1+\gamma^{\mathrm{A}}\bar{\gamma}^{\mathrm{A}}} \circ (\delta\gamma^{\mathrm{A}} - \gamma^{\mathrm{A}} \circ \delta\bar{\gamma}^{\mathrm{A}} \circ \gamma^{\mathrm{A}}) \circ \frac{1}{1+\gamma^{\mathrm{A}}\bar{\gamma}^{\mathrm{A}}}, \tag{2.20d}$$

with $\delta\gamma^{\mathrm{R,A}} = \delta\gamma^{\mathrm{R,A}}(\varepsilon, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, t)$. Equation for $\delta\gamma^{\mathrm{R,A}}$ is given as

$$\begin{aligned}\hbar\boldsymbol{v}_{\mathrm{F}} \cdot \left(\boldsymbol{\nabla} - i\frac{2e\boldsymbol{A}}{\hbar}\right)\delta\gamma^{\mathrm{R,A}} - i(e\boldsymbol{v}_{\mathrm{F}} \cdot \delta\boldsymbol{A} \circ \gamma^{\mathrm{R,A}} + \gamma^{\mathrm{R,A}} \circ e\boldsymbol{v}_{\mathrm{F}} \cdot \delta\boldsymbol{A}) - 2i\varepsilon\delta\gamma^{\mathrm{R,A}} \\ + \frac{\hbar}{2\tau}(\langle g_0^{\mathrm{R,A}}\rangle_{\mathrm{F}} \circ \delta\gamma^{\mathrm{R,A}} + \delta\gamma^{\mathrm{R,A}} \circ \langle \bar{g}_0^{\mathrm{R,A}}\rangle_{\mathrm{F}} + \langle \delta g_0^{\mathrm{R,A}}\rangle_{\mathrm{F}} \circ \gamma^{\mathrm{R,A}} + \gamma^{\mathrm{R,A}} \circ \langle \delta\bar{g}_0^{\mathrm{R,A}}\rangle_{\mathrm{F}}) \\ + \gamma^{\mathrm{R,A}}\left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_0^{\mathrm{R,A}}\rangle_{\mathrm{F}}\right) \circ \delta\gamma^{\mathrm{R,A}} + \delta\gamma^{\mathrm{R,A}} \circ \left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_0^{\mathrm{R,A}}\rangle_{\mathrm{F}}\right)\gamma^{\mathrm{R,A}} \\ + \gamma^{\mathrm{R,A}} \circ \left(\delta\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \delta\bar{f}_0^{\mathrm{R,A}}\rangle_{\mathrm{F}}\right) \circ \gamma^{\mathrm{R,A}} - \left(\delta\Delta\phi + \frac{\hbar}{2\tau}\langle \delta f_0^{\mathrm{R,A}}\rangle_{\mathrm{F}}\right) = 0,\end{aligned} \tag{2.21}$$

where $\delta g_0^{\rm A} = -\delta g_0^{\rm R*}$, $\delta f_0^{\rm A} = \delta \bar{f}_0^{\rm R*}$ and $\delta\gamma^{\rm A} = -\delta\bar{\gamma}^{\rm R*}$ are also obtained from the symmetry relation $\delta\hat{g}_0^{\rm A} = -\hat{\tau}_3\delta\hat{g}_0^{\rm R\dagger}\hat{\tau}_3$.

Let us perform the Fourier transform $\delta g_0^{\rm R}(\varepsilon,\boldsymbol{p}_{\rm F},\boldsymbol{r},t) = \delta g_0^{\rm R}(\varepsilon,\boldsymbol{p}_{\rm F},\boldsymbol{r},\omega){\rm e}^{-i\omega t}$ and introduce $g_{0\pm}^{\rm R,A}(\varepsilon,\boldsymbol{p}_{\rm F},\boldsymbol{r})$
$\equiv g_0^{\rm R,A}(\varepsilon \pm \hbar\omega/2,\boldsymbol{p}_{\rm F},\boldsymbol{r})$, $f_{0\pm}^{\rm R,A}(\varepsilon,\boldsymbol{p}_{\rm F},\boldsymbol{r}) \equiv f_0^{\rm R,A}(\varepsilon \pm \hbar\omega/2,\boldsymbol{p}_{\rm F},\boldsymbol{r})$ and $\gamma_\pm^{\rm R,A}(\varepsilon,\boldsymbol{p}_{\rm F},\boldsymbol{r}) \equiv \gamma^{\rm R,A}(\varepsilon \pm \hbar\omega/2,\boldsymbol{p}_{\rm F},\boldsymbol{r})$. Then we can express $\delta g_0^{\rm R,A} = \delta g_0^{\rm R,A}(\varepsilon,\boldsymbol{p}_{\rm F},\boldsymbol{r},\omega)$ and $\delta f_0^{\rm R,A} = \delta f_0^{\rm R,A}(\varepsilon,\boldsymbol{p}_{\rm F},\boldsymbol{r},\omega)$ alternatively as

$$\delta g_0^{\rm R} = -2\frac{\bar{\gamma}_-^{\rm R}\delta\gamma^{\rm R} + \gamma_+^{\rm R}\delta\bar{\gamma}^{\rm R}}{(1+\gamma_+^{\rm R}\bar{\gamma}_+^{\rm R})(1+\gamma_-^{\rm R}\bar{\gamma}_-^{\rm R})}, \tag{2.22a}$$
$$\delta g_0^{\rm A} = 2\frac{\bar{\gamma}_-^{\rm A}\delta\gamma^{\rm A} + \gamma_+^{\rm A}\delta\bar{\gamma}^{\rm A}}{(1+\gamma_+^{\rm A}\bar{\gamma}_+^{\rm A})(1+\gamma_-^{\rm A}\bar{\gamma}_-^{\rm A})}, \tag{2.22b}$$
$$\delta f_0^{\rm R} = 2\frac{\delta\gamma^{\rm R} - \gamma_+^{\rm R}\gamma_-^{\rm R}\delta\bar{\gamma}^{\rm R}}{(1+\gamma_+^{\rm R}\bar{\gamma}_+^{\rm R})(1+\gamma_-^{\rm R}\bar{\gamma}_-^{\rm R})}, \tag{2.22c}$$
$$\delta f_0^{\rm A} = -2\frac{\delta\gamma^{\rm A} - \gamma_+^{\rm A}\gamma_-^{\rm A}\delta\bar{\gamma}^{\rm A}}{(1+\gamma_+^{\rm A}\bar{\gamma}_+^{\rm A})(1+\gamma_-^{\rm A}\bar{\gamma}_-^{\rm A})}, \tag{2.22d}$$

where the barred functions are given generally by $\delta\bar{g}_0^{\rm R}(\varepsilon,\boldsymbol{p}_{\rm F},\boldsymbol{r},\omega) \equiv \delta g_0^{\rm R*}(-\varepsilon,-\boldsymbol{p}_{\rm F},\boldsymbol{r},-\omega)$. Equation for $\delta\gamma^{\rm R,A} = \delta\gamma^{\rm R,A}(\varepsilon,\boldsymbol{p}_{\rm F},\boldsymbol{r},\omega)$ is given as

$$\begin{aligned}
&\hbar\boldsymbol{v}_{\rm F}\cdot\left(\boldsymbol{\nabla} - i\frac{2e\boldsymbol{A}}{\hbar}\right)\delta\gamma^{\rm R,A} + 2\left[-i\varepsilon + \frac{\hbar}{4\tau}(\langle g_{0+}^{\rm R,A}\rangle_{\rm F} + \langle\bar{g}_{0-}^{\rm R,A}\rangle_{\rm F})\right]\delta\gamma^{\rm R,A}\\
&\quad - ie\boldsymbol{v}_{\rm F}\cdot\delta\boldsymbol{A}(\gamma_+^{\rm R,A} + \gamma_-^{\rm R,A}) + \frac{\hbar}{2\tau}(\langle\delta g_0^{\rm R,A}\rangle_{\rm F}\gamma_-^{\rm R,A} + \langle\delta\bar{g}_0^{\rm R,A}\rangle_{\rm F}\gamma_+^{\rm R,A})\\
&\quad + \left[\left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle\bar{f}_{0+}^{\rm R,A}\rangle_{\rm F}\right)\gamma_+^{\rm R,A} + \left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle\bar{f}_{0-}^{\rm R,A}\rangle_{\rm F}\right)\gamma_-^{\rm R,A}\right]\delta\gamma^{\rm R,A}\\
&\quad + \left(\delta\bar{\Delta}\phi^* + \frac{\hbar}{2\tau}\langle\delta\bar{f}_0^{\rm R,A}\rangle_{\rm F}\right)\gamma_+^{\rm R,A}\gamma_-^{\rm R,A} - \left(\delta\Delta\phi + \frac{\hbar}{2\tau}\langle\delta f_0^{\rm R,A}\rangle_{\rm F}\right) = 0.
\end{aligned} \tag{2.23}$$

We move on to consider the equations for the Keldysh Green's functions $\delta g_0^{\rm K}$ and $\delta f_0^{\rm K}$, which are relevant to the change in the distribution functions. Let us express $\delta\hat{g}_0^{\rm K}$ as

$$\delta\hat{g}_0^{\rm K} = \delta\hat{g}_0^{\rm R}\circ\tanh\frac{\varepsilon}{2k_{\rm B}T} - \tanh\frac{\varepsilon}{2k_{\rm B}T}\circ\delta\hat{g}_0^{\rm A} + \delta\hat{g}_0^{\rm a}, \tag{2.24}$$

with

$$\delta\hat{g}_0^{\rm a} = \begin{bmatrix} \delta g_0^{\rm a} & -i\delta f_0^{\rm a} \\ -i\delta\bar{f}_0^{\rm a} & \delta\bar{g}_0^{\rm a} \end{bmatrix}. \tag{2.25}$$

We also express $\delta g_0^{\rm a} = \delta g_0^{\rm a}(\varepsilon,\boldsymbol{p}_{\rm F},\boldsymbol{r},t)$ and $\delta f_0^{\rm a} = \delta f_0^{\rm a}(\varepsilon,\boldsymbol{p}_{\rm F},\boldsymbol{r},t)$ as

$$\delta g_0^{\rm a} = 2\frac{1}{1+\gamma^{\rm R}\bar{\gamma}^{\rm R}}\circ(\delta x^{\rm a} + \gamma^{\rm R}\circ\delta\bar{x}^{\rm a}\circ\bar{\gamma}^{\rm A})\circ\frac{1}{1+\gamma^{\rm A}\bar{\gamma}^{\rm A}}, \tag{2.26a}$$
$$\delta f_0^{\rm a} = 2\frac{1}{1+\gamma^{\rm R}\bar{\gamma}^{\rm R}}\circ(-\gamma^{\rm R}\circ\delta\bar{x}^{\rm a} + \delta x^{\rm a}\circ\gamma^{\rm A})\circ\frac{1}{1+\gamma^{\rm A}\bar{\gamma}^{\rm A}} \tag{2.26b}$$

with $\delta x^{\rm a} = \delta x^{\rm a}(\varepsilon, \boldsymbol{p}_{\rm F}, \boldsymbol{r}, t)$. Equation for $\delta x^{\rm a}$ is given as

$$
\begin{aligned}
&\hbar \boldsymbol{v}_{\rm F} \cdot \boldsymbol{\nabla} \delta x^{\rm a} + \hbar \frac{\partial \delta x^{\rm a}}{\partial t} + \frac{\hbar}{2\tau} (\langle g_0^{\rm R} \rangle_{\rm F} \circ \delta x^{\rm a} - \delta x^{\rm a} \circ \langle g_0^{\rm A} \rangle_{\rm F}) \\
&\quad - \frac{\hbar}{2\tau} (\langle \delta g_0^{\rm a} \rangle_{\rm F} + \gamma^{\rm R} \circ \langle \delta \bar{g}_0^{\rm a} \rangle_{\rm F} \circ \bar{\gamma}^{\rm A} + \langle \delta f_0^{\rm a} \rangle_{\rm F} \circ \bar{\gamma}^{\rm A} - \gamma^{\rm R} \circ \langle \delta \bar{f}_0^{\rm a} \rangle_{\rm F}) \\
&\quad - i \left(e \boldsymbol{v}_{\rm F} \cdot \delta \boldsymbol{A} \circ \tanh \frac{\varepsilon}{2k_{\rm B}T} - \tanh \frac{\varepsilon}{2k_{\rm B}T} \circ e \boldsymbol{v}_{\rm F} \cdot \delta \boldsymbol{A} \right) \\
&\quad + \gamma^{\rm R} \left(\Delta^* \phi^* + \frac{\hbar}{2\tau} \langle \bar{f}_0^{\rm R} \rangle_{\rm F} \right) \circ \delta x^{\rm a} - \delta x^{\rm a} \circ \left(\Delta \phi + \frac{\hbar}{2\tau} \langle f_0^{\rm A} \rangle_{\rm F} \right) \bar{\gamma}^{\rm A} \\
&\quad + \delta \Delta \phi \circ \tanh \frac{\varepsilon}{2k_{\rm B}T} \circ \bar{\gamma}^{\rm A} - \tanh \frac{\varepsilon}{2k_{\rm B}T} \circ \delta \Delta \phi \circ \bar{\gamma}^{\rm A} \\
&\quad + \gamma^{\rm R} \circ \delta \Delta^* \phi^* \circ \tanh \frac{\varepsilon}{2k_{\rm B}T} - \gamma^{\rm R} \circ \tanh \frac{\varepsilon}{2k_{\rm B}T} \circ \delta \Delta^* \phi^* = 0.
\end{aligned} \tag{2.27}
$$

Performing the Fourier transform, we obtain $\delta g_0^{\rm a} = \delta g_0^{\rm a}(\varepsilon, \boldsymbol{p}_{\rm F}, \boldsymbol{r}, \omega)$ and $\delta f_0^{\rm a} = \delta f_0^{\rm a}(\varepsilon, \boldsymbol{p}_{\rm F}, \boldsymbol{r}, \omega)$ as

$$
\delta g_0^{\rm a} = 2 \frac{\delta x^{\rm a} + \gamma_+^{\rm R} \bar{\gamma}_-^{\rm A} \delta \bar{x}^{\rm a}}{(1 + \gamma_+^{\rm R} \bar{\gamma}_+^{\rm R})(1 + \gamma_-^{\rm A} \bar{\gamma}_-^{\rm A})}, \tag{2.28a}
$$

$$
\delta f_0^{\rm a} = 2 \frac{\gamma_-^{\rm A} \delta x^{\rm a} - \gamma_+^{\rm R} \delta \bar{x}^{\rm a}}{(1 + \gamma_+^{\rm R} \bar{\gamma}_+^{\rm R})(1 + \gamma_-^{\rm A} \bar{\gamma}_-^{\rm A})}. \tag{2.28b}
$$

Functions $\delta g_0^{\rm K} = \delta g_0^{\rm K}(\varepsilon, \boldsymbol{p}_{\rm F}, \boldsymbol{r}, \omega)$ and $\delta f_0^{\rm K} = \delta f_0^{\rm K}(\varepsilon, \boldsymbol{p}_{\rm F}, \boldsymbol{r}, \omega)$ are also Fourier-transformed into

$$
\delta g_0^{\rm K} = \delta g_0^{\rm R} \tanh \frac{\varepsilon_-}{2k_{\rm B}T} - \delta g_0^{\rm A} \tanh \frac{\varepsilon_+}{2k_{\rm B}T} + \delta g_0^{\rm a}, \tag{2.29a}
$$

$$
\delta f_0^{\rm K} = \delta f_0^{\rm R} \tanh \frac{\varepsilon_-}{2k_{\rm B}T} - \delta f_0^{\rm A} \tanh \frac{\varepsilon_+}{2k_{\rm B}T} + \delta f_0^{\rm a}, \tag{2.29b}
$$

with $\varepsilon_\pm \equiv \varepsilon \pm \hbar\omega/2$. Equation for $\delta x^{\rm a} = \delta x^{\rm a}(\varepsilon, \boldsymbol{p}_{\rm F}, \boldsymbol{r}, \omega)$ is obtained from Eq. (2.27) in the time domain as

$$
\begin{aligned}
&\hbar \boldsymbol{v}_{\rm F} \cdot \boldsymbol{\nabla} \delta x^{\rm a} - i\hbar\omega \delta x^{\rm a} + \frac{\hbar}{2\tau} (\langle g_{0+}^{\rm R} \rangle_{\rm F} - \langle g_{0-}^{\rm A} \rangle_{\rm F}) \delta x^{\rm a} \\
&\quad - \frac{\hbar}{2\tau} (\langle \delta g_0^{\rm a} \rangle_{\rm F} + \langle \delta \bar{g}_0^{\rm a} \rangle_{\rm F} \gamma_+^{\rm R} \bar{\gamma}_-^{\rm A} + \langle \delta f_0^{\rm a} \rangle_{\rm F} \bar{\gamma}_-^{\rm A} - \langle \delta \bar{f}_0^{\rm a} \rangle_{\rm F} \gamma_+^{\rm R}) \\
&\quad + \left[\left(\Delta^* \phi^* + \frac{\hbar}{2\tau} \langle \bar{f}_{0+}^{\rm R} \rangle_{\rm F} \right) \gamma_+^{\rm R} - \left(\Delta \phi + \frac{\hbar}{2\tau} \langle f_{0-}^{\rm A} \rangle_{\rm F} \right) \bar{\gamma}_-^{\rm A} \right] \delta x^{\rm a} \\
&\quad + \left(\tanh \frac{\varepsilon_-}{2k_{\rm B}T} - \tanh \frac{\varepsilon_+}{2k_{\rm B}T} \right) (\delta \bar{\Delta} \phi^* \gamma_+^{\rm R} + \delta \Delta \phi \bar{\gamma}_-^{\rm A} - ie \boldsymbol{v}_{\rm F} \cdot \delta \boldsymbol{A}) \\
&\quad = 0.
\end{aligned} \tag{2.30}
$$

2.4 Equation for the Hall electric field

We now consider Eqs. (2.16b) and (2.17b) that describes the Hall effect for clean superconductors $\hbar/\tau\Delta_0 \ll 1$ with keeping the limit $\omega \to 0$ in mind. Hence, we can neglect the time derivatives and impurity self-energy terms in Eqs. (2.16b) and (2.17b) to an excellent approximation. On the other hand, although this approximation is sufficient to describe the Hall effect, it is not possible to describe the spatial variation of the pair potential due to the Hall electric field because $\delta f_1^{\mathrm{R}} = 0$ and $\delta f_1^{\mathrm{K}} = 0$ in this approximation. We need to consider the first-order of ω/Δ_0 and $\hbar/\tau\Delta_0$ to describe spatial variations of pair potential.

Removing the impurity self-energy terms in Eq. (2.16b), we obtain the equation for $\hat{g}_1^{\mathrm{K}}$ as

$$\left[\varepsilon\hat{\tau}_3 - \hat{\Delta}, \hat{g}_1^{\mathrm{K}}\right] + i\hbar\boldsymbol{v}_{\mathrm{F}}\cdot\boldsymbol{\partial}\hat{g}_1^{\mathrm{K}} + \frac{i\hbar}{2}\left[e\boldsymbol{v}_{\mathrm{F}}\cdot\boldsymbol{E}_1\frac{\partial}{\partial\varepsilon} + e(\boldsymbol{v}_{\mathrm{F}}\times\boldsymbol{B})\cdot\frac{\partial}{\partial\boldsymbol{p}_{\mathrm{F}}}\right]\{\hat{\tau}_3, \hat{g}_0^{\mathrm{K}}\} = \hat{0}. \tag{2.31}$$

The $(1,1)$ and $(1,2)$ elements of the equation for $\hat{g}_1^{\mathrm{K}}$ are given as

$$\hbar\boldsymbol{v}_{\mathrm{F}}\cdot\boldsymbol{\nabla}g_1^{\mathrm{K}} + \hbar e\boldsymbol{v}_{\mathrm{F}}\cdot\boldsymbol{E}_1\frac{\partial g_0^{\mathrm{K}}}{\partial\varepsilon} + \hbar e(\boldsymbol{v}_{\mathrm{F}}\times\boldsymbol{B})\cdot\frac{\partial g_0^{\mathrm{K}}}{\partial\boldsymbol{p}_{\mathrm{F}}} - \Delta\phi\bar{f}_1^{\mathrm{K}} - \Delta^*\phi^* f_1^{\mathrm{K}} = 0, \tag{2.32a}$$

$$2\varepsilon_n f_1^{\mathrm{K}} + \hbar\boldsymbol{v}_{\mathrm{F}}\cdot\left(\boldsymbol{\nabla} - i\frac{2e\boldsymbol{A}}{\hbar}\right)f_1^{\mathrm{K}} + \Delta\phi\bar{g}_1^{\mathrm{K}} - \Delta\phi g_1^{\mathrm{K}} = 0. \tag{2.32b}$$

The $(2,2)$ and $(2,1)$ elements are obtained from above by setting $(\varepsilon, \boldsymbol{p}_{\mathrm{F}}) \to (-\varepsilon, -\boldsymbol{p}_{\mathrm{F}})$, taking the complex conjugate, and keeping Eq. (2.3) in mind. We then write the gradient term in the $(1,1)$ and $(2,2)$ elements together with the electric-field term as

$$\boldsymbol{\nabla}g_1^{\mathrm{K}\prime} \equiv \boldsymbol{\nabla}g_1^{\mathrm{K}} + e\boldsymbol{E}_1\frac{\partial g_0^{\mathrm{K}}}{\partial\varepsilon}, \tag{2.33a}$$

The four equations for $g_1^{\mathrm{K}\prime}$, $\bar{g}_1^{\mathrm{K}\prime}$, f_1^{K} and $\bar{f}_1^{\mathrm{K}}$ can be reduced to a single equation for $g_1^{\mathrm{K}\prime}$ as follows: Using $\bar{g}_0^{\mathrm{K}} = -g_0^{\mathrm{K}}$, and writing them in terms of $g_1^{\mathrm{K}\prime} + \bar{g}_1^{\mathrm{K}\prime}$ and $g_1^{\mathrm{K}\prime} - \bar{g}_1^{\mathrm{K}\prime}$, we obtain the four equations for $g_1^{\mathrm{K}\prime} + \bar{g}_1^{\mathrm{K}\prime}$, $g_1^{\mathrm{K}\prime} - \bar{g}_1^{\mathrm{K}\prime}$, f_1^{K} and $\bar{f}_1^{\mathrm{K}}$:

$$\hbar\boldsymbol{v}_{\mathrm{F}}\cdot\boldsymbol{\nabla}\frac{g_1^{\mathrm{K}\prime} + \bar{g}_1^{\mathrm{K}\prime}}{2} + \hbar e(\boldsymbol{v}_{\mathrm{F}}\times\boldsymbol{B})\cdot\frac{\partial g_0^{\mathrm{K}}}{\partial\boldsymbol{p}_{\mathrm{F}}} = 0, \tag{2.34a}$$

$$\hbar\boldsymbol{v}_{\mathrm{F}}\cdot\boldsymbol{\nabla}\frac{g_1^{\mathrm{K}\prime} - \bar{g}_1^{\mathrm{K}\prime}}{2} - \Delta\phi\bar{f}_1^{\mathrm{K}} - \Delta^*\phi^* f_1^{\mathrm{K}} = 0, \tag{2.34b}$$

$$\varepsilon_n f_1^{\mathrm{K}} + \frac{\hbar\boldsymbol{v}_{\mathrm{F}}}{2}\cdot\left(\boldsymbol{\nabla} - i\frac{2e\boldsymbol{A}}{\hbar}\right)f_1^{\mathrm{K}} - \Delta\phi\frac{g_1^{\mathrm{K}\prime} - \bar{g}_1^{\mathrm{K}\prime}}{2} = 0, \tag{2.34c}$$

$$\varepsilon_n\bar{f}_1^{\mathrm{K}} - \frac{\hbar\boldsymbol{v}_{\mathrm{F}}}{2}\cdot\left(\boldsymbol{\nabla} + i\frac{2e\boldsymbol{A}}{\hbar}\right)\bar{f}_1^{\mathrm{K}} + \Delta^*\phi^*\frac{g_1^{\mathrm{K}\prime} - \bar{g}_1^{\mathrm{K}\prime}}{2} = 0. \tag{2.34d}$$

Equations (2.34b), (2.34c) and (2.34d) are linear closed equations without the external source. We hence conclude $f_1^{\mathrm{K}} = 0$, $g_1^{\mathrm{K}\prime} = \bar{g}_1^{\mathrm{K}\prime}$. Substitution of this result into the equation for $g_1^{\mathrm{K}\prime} + \bar{g}_1^{\mathrm{K}\prime}$ yields

$$\boldsymbol{v}_{\mathrm{F}}\cdot\boldsymbol{\nabla}g_1^{\mathrm{K}\prime} + e(\boldsymbol{v}_{\mathrm{F}}\times\boldsymbol{B})\cdot\frac{\partial g_0^{\mathrm{K}}}{\partial\boldsymbol{p}_{\mathrm{F}}} = 0. \tag{2.35}$$

Equation (2.17b) can be simplified similarly. Removing the the time-derivatives and impurity self-energy terms in Eq. (2.17b), the $(1,1)$ and $(1,2)$ elements of the equations for $\delta\hat{g}_1^{\mathrm{K}}$ are given as

$$\begin{aligned}
&\hbar \boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\nabla} \delta g_1^{\mathrm{K}} + \hbar e \boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{E}_1 \frac{\partial \delta g_0^{\mathrm{K}}}{\partial \varepsilon} + \hbar e (\boldsymbol{v}_{\mathrm{F}} \times \boldsymbol{B}) \cdot \frac{\partial \delta g_0^{\mathrm{K}}}{\partial \boldsymbol{p}_{\mathrm{F}}} \\
&\quad + \hbar e \boldsymbol{v}_{\mathrm{F}} \cdot \delta \boldsymbol{E}_1 \frac{\partial g_0^{\mathrm{K}}}{\partial \varepsilon} + \hbar e (\boldsymbol{v}_{\mathrm{F}} \times \delta \boldsymbol{B}) \cdot \frac{\partial g_0^{\mathrm{K}}}{\partial \boldsymbol{p}_{\mathrm{F}}} + \hbar e \boldsymbol{v}_{\mathrm{F}} \cdot \delta \boldsymbol{E}_0 \frac{\partial g_1^{\mathrm{K}}}{\partial \varepsilon} \\
&\quad - \Delta \phi \delta \bar{f}_1^{\mathrm{K}} - \Delta^* \phi^* \delta f_1^{\mathrm{K}} - \delta \Delta \phi \bar{f}_1^{\mathrm{K}} - \delta \Delta^* \phi^* f_1^{\mathrm{K}} = 0,
\end{aligned} \tag{2.36a}$$

$$\begin{aligned}
&2\varepsilon_n \delta f_1^{\mathrm{K}} + \hbar \boldsymbol{v}_{\mathrm{F}} \cdot \left(\boldsymbol{\nabla} - i \frac{2e\boldsymbol{A}}{\hbar} \right) \delta f_1^{\mathrm{K}} - 2e \boldsymbol{v}_{\mathrm{F}} \cdot \delta \boldsymbol{A} f_1^{\mathrm{K}} \\
&\quad + \Delta \phi \delta \bar{g}_1^{\mathrm{K}} - \Delta \phi \delta g_1^{\mathrm{K}} + \delta \Delta \phi \bar{g}_1^{\mathrm{K}} - \delta \Delta \phi g_1^{\mathrm{K}} = 0.
\end{aligned} \tag{2.36b}$$

Let us write the gradient term in Eq. (2.36a) together with the electric-field term as

$$\boldsymbol{\nabla} \delta g_1^{\mathrm{K}\prime} \equiv \boldsymbol{\nabla} \delta g_1^{\mathrm{K}} + e \boldsymbol{E}_1 \frac{\partial \delta g_0^{\mathrm{K}}}{\partial \varepsilon} + e \delta \boldsymbol{E}_0 \frac{\partial g_1^{\mathrm{K}}}{\partial \varepsilon} + e \delta \boldsymbol{E}_1 \frac{\partial g_0^{\mathrm{K}}}{\partial \varepsilon}, \tag{2.37}$$

and use $\bar{g}_0^{\mathrm{K}} = -g_0^{\mathrm{K}}$, $f_1^{\mathrm{K}} = 0$ and $\bar{g}_1^{\mathrm{K}} = g_1^{\mathrm{K}}$ obtained from $\bar{g}_1^{\mathrm{K}\prime} = g_1^{\mathrm{K}\prime}$. We then obtain the four equations for $\delta g_1^{\mathrm{K}\prime} + \delta \bar{g}_1^{\mathrm{K}\prime}$, $\delta g_1^{\mathrm{K}\prime} - \delta \bar{g}_1^{\mathrm{K}\prime}$, δf_1^{K} and $\delta \bar{f}_1^{\mathrm{K}}$:

$$\hbar \boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\nabla} \frac{\delta g_1^{\mathrm{K}\prime} + \delta \bar{g}_1^{\mathrm{K}\prime}}{2} + \hbar e (\boldsymbol{v}_{\mathrm{F}} \times \boldsymbol{B}) \cdot \frac{\partial \delta g_0^{\mathrm{K}}}{\partial \boldsymbol{p}_{\mathrm{F}}} + \hbar e (\boldsymbol{v}_{\mathrm{F}} \times \delta \boldsymbol{B}) \cdot \frac{\partial g_0^{\mathrm{K}}}{\partial \boldsymbol{p}_{\mathrm{F}}} = 0, \tag{2.38a}$$

$$\hbar \boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\nabla} \frac{\delta g_1^{\mathrm{K}\prime} - \delta \bar{g}_1^{\mathrm{K}\prime}}{2} - \Delta \phi \delta \bar{f}_1^{\mathrm{K}} - \Delta^* \phi^* \delta f_1^{\mathrm{K}} = 0, \tag{2.38b}$$

$$\varepsilon_n \delta f_1^{\mathrm{K}} + \frac{\hbar \boldsymbol{v}_{\mathrm{F}}}{2} \cdot \left(\boldsymbol{\nabla} - i \frac{2e\boldsymbol{A}}{\hbar} \right) \delta f_1^{\mathrm{K}} - \Delta \phi \frac{\delta g_1^{\mathrm{K}\prime} - \delta \bar{g}_1^{\mathrm{K}\prime}}{2} = 0, \tag{2.38c}$$

$$\varepsilon_n \delta \bar{f}_1^{\mathrm{K}} - \frac{\hbar \boldsymbol{v}_{\mathrm{F}}}{2} \cdot \left(\boldsymbol{\nabla} + i \frac{2e\boldsymbol{A}}{\hbar} \right) \delta \bar{f}_1^{\mathrm{K}} + \Delta^* \phi^* \frac{\delta g_1^{\mathrm{K}\prime} - \delta \bar{g}_1^{\mathrm{K}\prime}}{2} = 0. \tag{2.38d}$$

Equations (2.38b), (2.38c) and (2.38d) are linear closed equations without the external source. We hence conclude $\delta f_1^{\mathrm{K}} = 0$, $\delta g_1^{\mathrm{K}\prime} = \delta \bar{g}_1^{\mathrm{K}\prime}$. Substitution of this result into the equation for $\delta g_1^{\mathrm{K}\prime} + \delta \bar{g}_1^{\mathrm{K}\prime}$ yields

$$\boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\nabla} \delta g_1^{\mathrm{K}\prime} + e (\boldsymbol{v}_{\mathrm{F}} \times \boldsymbol{B}) \cdot \frac{\partial \delta g_0^{\mathrm{K}}}{\partial \boldsymbol{p}_{\mathrm{F}}} + e (\boldsymbol{v}_{\mathrm{F}} \times \delta \boldsymbol{B}) \cdot \frac{\partial g_0^{\mathrm{K}}}{\partial \boldsymbol{p}_{\mathrm{F}}} = 0. \tag{2.39}$$

We start from these expressions to obtain the equations for the Hall electric field: First, the charge densities originating from g_1^{K} and δg_1^{K} are given by

$$\rho_1 = -\frac{eN(0)}{2} \int_{-\infty}^{\infty} \langle g_1^{\mathrm{K}} \rangle_{\mathrm{F}} d\varepsilon, \tag{2.40a}$$

$$\delta \rho_1 = -\frac{eN(0)}{2} \int_{-\infty}^{\infty} \langle \delta g_1^{\mathrm{K}} \rangle_{\mathrm{F}} d\varepsilon. \tag{2.40b}$$

Let us apply the operator $\boldsymbol{\nabla}$ to these equations, substitute Eqs. (2.33a) and (2.37), and use $g_0^{\rm K} \to \pm 2$, $\delta g_0^{\rm K} \to 0$ and $g_1^{\rm K} \to 0$ for $\varepsilon \to \pm\infty$ to perform integration with respect to ε for the electric-field term. The procedure yields

$$\boldsymbol{\nabla}\rho_1 = -\frac{eN(0)}{2}\boldsymbol{\nabla}\int_{-\infty}^{\infty}\langle g_1^{\rm K\prime}\rangle_{\rm F} d\varepsilon + 2e^2N(0)\boldsymbol{E}_1 \tag{2.41a}$$

$$\boldsymbol{\nabla}\delta\rho_1 = -\frac{eN(0)}{2}\boldsymbol{\nabla}\int_{-\infty}^{\infty}\langle \delta g_1^{\rm K}\rangle_{\rm F} d\varepsilon + 2e^2N(0)\delta\boldsymbol{E}_1 \tag{2.41b}$$

We then substitute Eqs. (2.33a) and (2.37) and Gauss's law $\rho = \epsilon_0\boldsymbol{\nabla}\cdot\boldsymbol{E}$, and use identities $\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{E} = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E} + \boldsymbol{\nabla}^2\boldsymbol{E}$ and $\boldsymbol{\nabla}\times\boldsymbol{E}_1 = \boldsymbol{0}$. We thereby obtain

$$-\lambda_{\rm TF}^2\boldsymbol{\nabla}^2\boldsymbol{E}_1 + \boldsymbol{E}_1 = \frac{1}{4e}\boldsymbol{\nabla}\int_{-\infty}^{\infty}\langle g_1^{\rm K\prime}\rangle_{\rm F} d\varepsilon, \tag{2.42a}$$

$$-\lambda_{\rm TF}^2(\boldsymbol{\nabla}^2\delta\boldsymbol{E}_1 + \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\delta\boldsymbol{E}_1) + \delta\boldsymbol{E}_1 = \frac{1}{4e}\boldsymbol{\nabla}\int_{-\infty}^{\infty}\langle \delta g_1^{\rm K\prime}\rangle_{\rm F} d\varepsilon, \tag{2.42b}$$

where $\lambda_{\rm TF} \equiv \sqrt{\epsilon_0/2e^2N(0)}$ is the Thomas–Fermi screening length. Since the electric-field screening length $\lambda_{\rm TF} \sim k_{\rm F}^{-1}$ is short, we can neglect the gradient terms for the electric field in a case where the spatial variation of the electric field is small. Equations (2.35) and (2.42a) enable us to calculate the Hall electric field due to equilibrium supercurrent microscopically based on the solution of the standard Eilenberger equations, and Eqs. (2.39) and (2.42b) enable us to calculate the Hall electric field in the flux-flow state.

2.5 Analytic continuation in terms of energy

We now transform the energy variable ε in the Keldysh formalism into the Matsubara energy $\varepsilon_n \equiv \pi k_{\rm B}T(2n+1)$ on the imaginary axis so as to perform the numerical calculations efficiently.

We start from the current density $\boldsymbol{j}$ and pair potential Δ for the zeroth-order response:

$$\boldsymbol{j} = -\frac{eN(0)}{2}\int_{-\infty}^{\infty}\left\langle \boldsymbol{v}_{\rm F} g_0^{\rm K}\right\rangle_{\rm F} d\varepsilon, \tag{2.43a}$$

$$\Delta = \frac{g_0}{4i}\int_{-\infty}^{\infty}\left\langle f_0^{\rm K}\phi^*\right\rangle_{\rm F} d\varepsilon. \tag{2.43b}$$

Using $\hat{g}_0^{\rm K} = (\hat{g}_0^{\rm R} - \hat{g}_0^{\rm A})\tanh(\varepsilon/2k_{\rm B}T)$ and $\hat{g}_0^{\rm A} = -\hat{\tau}_3\hat{g}_0^{\rm R\dagger}\hat{\tau}_3$, we then obtain the expressions for the current density and pair potential as

$$\boldsymbol{j} = -\frac{eN(0)}{2}\int_{-\infty}^{\infty} d\varepsilon \left\langle \boldsymbol{v}_{\rm F}(g_0^{\rm R} + g_0^{\rm R*})\right\rangle_{\rm F} \tanh\frac{\varepsilon}{2k_{\rm B}T}, \tag{2.44a}$$

$$\Delta = \frac{g_0}{4i}\int_{-\infty}^{\infty} d\varepsilon \left\langle (f_0^{\rm R} - \bar{f}_0^{\rm R*})\phi^*\right\rangle_{\rm F} \tanh\frac{\varepsilon}{2k_{\rm B}T}. \tag{2.44b}$$

The integration path in Eq. (2.44) running from $-\infty$ to ∞ can be closed in the upper half of the complex ε plane by adding an infinite semi-circle path with null contribution.

Within the contour, the integrand has poles at $i\varepsilon_n$ with the residue $2k_{\rm B}T$ that originate from $\tanh(\varepsilon/2k_{\rm B}T)$. Collecting their contributions, we can express the current density and pair potential in terms of $g_0(\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r}) \equiv g_0^{\rm R}(i\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r})$, $f_0(\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r}) \equiv f_0^{\rm R}(i\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r})$ and $\bar{f}_0(\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r}) \equiv \bar{f}_0^{\rm R}(i\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r})$ as

$$\boldsymbol{j} = -2\pi i k_{\rm B}TeN(0)\sum_{n=0}^{\infty}\langle \boldsymbol{v}_{\rm F}(g_0 - g_0^*)\rangle_{\rm F}\,, \tag{2.45a}$$

$$\Delta = \pi \mathrm{g}_0 k_{\rm B}T\sum_{n=0}^{\infty}\left\langle (f_0 + \bar{f}_0^*)\phi^*\right\rangle_{\rm F}. \tag{2.45b}$$

The coupling constant (2.10) is rewritten as

$$\frac{1}{\mathrm{g}_0} = \ln\frac{T}{T_{\rm c}} + 2\pi k_{\rm B}T\sum_{n=0}^{\infty}\frac{1}{\varepsilon_n}. \tag{2.46}$$

Functions g_0 and f_0 are expressible as

$$g_0 = \frac{1-\gamma\bar{\gamma}}{1+\gamma\bar{\gamma}}, \tag{2.47a}$$

$$f_0 = \frac{2\gamma}{1+\gamma\bar{\gamma}}, \tag{2.47b}$$

with $\gamma(\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r}) \equiv \gamma^{\rm R}(i\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r})$ and $\bar{\gamma}(\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r}) \equiv \bar{\gamma}^{\rm R}(i\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r})$. Using $g_0 = \bar{g}_0$ given by $g_0^{\rm R} = \bar{g}_0^{\rm R}$, equation for γ is obtained from Eq. (2.19) as

$$\begin{aligned}\hbar\boldsymbol{v}_{\rm F}\cdot\left(\boldsymbol{\nabla} - i\frac{2e\boldsymbol{A}}{\hbar}\right)\gamma + 2\left(\varepsilon_n + \frac{\hbar}{2\tau}\langle g_0\rangle_{\rm F}\right)\gamma \\ + \left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_0\rangle_{\rm F}\right)\gamma^2 - \left(\Delta\phi + \frac{\hbar}{2\tau}\langle f_0\rangle_{\rm F}\right) = 0.\end{aligned} \tag{2.48}$$

Next, we focus on the charge density $\delta\rho_0$, current density $\delta\boldsymbol{j}$, and pair potential $\delta\Delta$ for the first-order response given by

$$\delta\rho_0 = -\frac{eN(0)}{2}\int_{-\infty}^{\infty}\langle \delta g_0^{\rm K}\rangle_{\rm F} d\varepsilon, \tag{2.49a}$$

$$\delta\boldsymbol{j} = -\frac{eN(0)}{2}\int_{-\infty}^{\infty}\langle \boldsymbol{v}_{\rm F}\delta g_0^{\rm K}\rangle_{\rm F} d\varepsilon, \tag{2.49b}$$

$$\delta\Delta = \frac{\mathrm{g}_0}{4i}\int_{-\infty}^{\infty}\left\langle \delta f_0^{\rm K}\phi^*\right\rangle_{\rm F} d\varepsilon. \tag{2.49c}$$

Let us express $\delta g_0^{\rm K}$ and $\delta f_0^{\rm K}$ in Eqs. (2.29a) and (2.29b) in terms of

$$\delta g_0^{\rm a\prime} \equiv \delta g_0^{\rm a}\left(\tanh\frac{\varepsilon_-}{2k_{\rm B}T} - \tanh\frac{\varepsilon_+}{2k_{\rm B}T}\right)^{-1}, \tag{2.50a}$$

$$\delta f_0^{\rm a\prime} \equiv \delta f_0^{\rm a}\left(\tanh\frac{\varepsilon_-}{2k_{\rm B}T} - \tanh\frac{\varepsilon_+}{2k_{\rm B}T}\right)^{-1}. \tag{2.50b}$$

as

$$\delta g_0^{\mathrm{K}} = \delta g_0^{\mathrm{R}} \tanh\frac{\varepsilon_-}{2k_{\mathrm{B}}T} - \delta g_0^{\mathrm{A}} \tanh\frac{\varepsilon_+}{2k_{\mathrm{B}}T} + \delta g_0^{\mathrm{a}\prime}\left(\tanh\frac{\varepsilon_-}{2k_{\mathrm{B}}T} - \tanh\frac{\varepsilon_+}{2k_{\mathrm{B}}T}\right), \tag{2.51a}$$

$$\delta f_0^{\mathrm{K}} = \delta f_0^{\mathrm{R}} \tanh\frac{\varepsilon_-}{2k_{\mathrm{B}}T} - \delta f_0^{\mathrm{A}} \tanh\frac{\varepsilon_+}{2k_{\mathrm{B}}T} + \delta f_0^{\mathrm{a}\prime}\left(\tanh\frac{\varepsilon_-}{2k_{\mathrm{B}}T} - \tanh\frac{\varepsilon_+}{2k_{\mathrm{B}}T}\right). \tag{2.51b}$$

We substitute Eq. (2.51a) into Eq. (2.49a) and use $\delta g_0^{\mathrm{R}} \to 0$ for $\varepsilon \to \pm\infty$ and $\delta g_0^{\mathrm{A}} = -\delta g_0^{\mathrm{R}*}$ to obtain

$$\begin{aligned}\delta\rho_0 = &-\frac{eN(0)}{2}\int_{-\infty}^{\infty} d\varepsilon \left\langle \delta g_{0+}^{\mathrm{R}} + \delta g_{0-}^{\mathrm{R}*} + \delta g_{0+}^{\mathrm{a}\prime} - \delta g_{0-}^{\mathrm{a}\prime}\right\rangle_{\mathrm{F}} \tanh\frac{\varepsilon}{2k_{\mathrm{B}}T}\\ &+ \frac{eN(0)}{4}\left\langle \delta g_{0+}^{\mathrm{a}\prime}(\varepsilon\to\infty) + \delta g_{0-}^{\mathrm{a}\prime}(\varepsilon\to\infty) + \delta g_{0+}^{\mathrm{a}\prime}(\varepsilon\to-\infty) + \delta g_{0-}^{\mathrm{a}\prime}(\varepsilon\to-\infty)\right\rangle_{\mathrm{F}}.\end{aligned} \tag{2.52}$$

We now calculate the second term on the right-hand side of Eq. (2.52). Equation for δg_0^{a} is given by

$$\begin{aligned}&- i\hbar\omega\delta g_0^{\mathrm{a}} + \hbar\boldsymbol{v}_{\mathrm{F}}\cdot\boldsymbol{\nabla}\delta g_0^{\mathrm{a}} + \frac{\hbar}{2\tau}(\langle g_{0+}^{\mathrm{R}}\rangle_{\mathrm{F}} - \langle g_{0-}^{\mathrm{A}}\rangle_{\mathrm{F}})\delta g_0^{\mathrm{a}} - \frac{\hbar}{2\tau}(g_{0+}^{\mathrm{R}} - g_{0-}^{\mathrm{A}})\langle\delta g_0^{\mathrm{a}}\rangle_{\mathrm{F}}\\ &\quad - \left(\Delta\phi + \frac{\hbar}{2\tau}\langle f_{0+}^{\mathrm{R}}\rangle_{\mathrm{F}}\right)\delta\bar{f}_0^{\mathrm{a}} - \left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle\bar{f}_{0-}^{\mathrm{A}}\rangle_{\mathrm{F}}\right)\delta f_0^{\mathrm{a}} + \frac{\hbar}{2\tau}(\langle\delta f_0^{\mathrm{a}}\rangle_{\mathrm{F}}\bar{f}_{0-}^{\mathrm{A}} + \langle\delta\bar{f}_0^{\mathrm{a}}\rangle_{\mathrm{F}} f_{0+}^{\mathrm{R}})\\ &\quad - \left(\tanh\frac{\varepsilon_-}{2k_{\mathrm{B}}T} - \tanh\frac{\varepsilon_+}{2k_{\mathrm{B}}T}\right)\left[\delta\Delta\phi\bar{f}_{0-}^{\mathrm{A}} - \delta\Delta^*\phi^* f_{0+}^{\mathrm{R}} + ie\boldsymbol{v}_{\mathrm{F}}\cdot\delta\boldsymbol{A}(g_{0+}^{\mathrm{R}} - g_{0-}^{\mathrm{A}})\right] = 0.\end{aligned} \tag{2.53}$$

Substituting Eqs. (2.50a) and (2.50b) into Eq. (2.53), we obtain equation for $\delta g_0^{\mathrm{a}\prime}$ as

$$\begin{aligned}&- i\hbar\omega\delta g_0^{\mathrm{a}\prime} + \hbar\boldsymbol{v}_{\mathrm{F}}\cdot\boldsymbol{\nabla}\delta g_0^{\mathrm{a}\prime} + \frac{\hbar}{2\tau}(\langle g_{0+}^{\mathrm{R}}\rangle_{\mathrm{F}} - \langle g_{0-}^{\mathrm{A}}\rangle_{\mathrm{F}})\delta g_0^{\mathrm{a}\prime} - \frac{\hbar}{2\tau}(g_{0+}^{\mathrm{R}} - g_{0-}^{\mathrm{A}})\langle\delta g_0^{\mathrm{a}\prime}\rangle_{\mathrm{F}}\\ &\quad + \left(\Delta\phi + \frac{\hbar}{2\tau}\langle f_{0+}^{\mathrm{R}}\rangle_{\mathrm{F}}\right)\delta\bar{f}_0^{\mathrm{a}\prime} + \left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle\bar{f}_{0-}^{\mathrm{A}}\rangle_{\mathrm{F}}\right)\delta f_0^{\mathrm{a}\prime} + \frac{\hbar}{2\tau}(\langle\delta f_0^{\mathrm{a}\prime}\rangle_{\mathrm{F}}\bar{f}_{0-}^{\mathrm{A}} - \langle\delta\bar{f}_0^{\mathrm{a}\prime}\rangle_{\mathrm{F}} f_{0+}^{\mathrm{R}})\\ &\quad - \delta\Delta\phi\bar{f}_{0-}^{\mathrm{A}} + \delta\Delta^*\phi^* f_{0+}^{\mathrm{R}} - ie\boldsymbol{v}_{\mathrm{F}}\cdot\delta\boldsymbol{A}(g_{0+}^{\mathrm{R}} - g_{0-}^{\mathrm{A}}) = 0.\end{aligned} \tag{2.54}$$

Let us use $g_0^{\mathrm{R}} \to 1$, $g_0^{\mathrm{A}} \to -1$ and $f_0^{\mathrm{R,A}} \to 0$ for $\varepsilon \to \pm\infty$, and approximate $\boldsymbol{\nabla}(\boldsymbol{v}_{\mathrm{F}}\cdot\delta\boldsymbol{E}) \approx \boldsymbol{0}$. We then obtain $\delta g_0^{\mathrm{a}\prime}$ for $\varepsilon \to \pm\infty$ as

$$\delta g_0^{\mathrm{a}\prime} \approx 2\frac{i\omega\tau}{1 - i\omega\tau}e\boldsymbol{v}_{\mathrm{F}}\cdot\delta\boldsymbol{A}. \tag{2.55}$$

Substituting Eq. (2.55) into Eq. (2.52), we observe that the second term on the right-hand side of Eq. (2.52) vanish because $\langle\boldsymbol{v}_{\mathrm{F}}\rangle_{\mathrm{F}} = \boldsymbol{0}$. We thereby obtain

$$\delta\rho_0 = -\frac{eN(0)}{2}\int_{-\infty}^{\infty} d\varepsilon \left\langle \delta g_{0+}^{\mathrm{R}} + \delta g_{0-}^{\mathrm{R}*} + \delta g_{0+}^{\mathrm{a}\prime} - \delta g_{0-}^{\mathrm{a}\prime}\right\rangle_{\mathrm{F}} \tanh\frac{\varepsilon}{2k_{\mathrm{B}}T}. \tag{2.56}$$

We also close this integration path over $-\infty \leq \varepsilon \leq \infty$ in the upper half of the complex ε plane and use the residue theorem subsequently to collect the residues in the closed path. We can thereby express the charge density in terms of $\delta g_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) \equiv \delta g^{\mathrm{R}}_{0\pm}(i\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ and $\delta g^{\mathrm{a}}_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) \equiv \delta g^{\mathrm{a}\prime}_{0\pm}(i\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ as

$$\delta\rho_0 = -2\pi i k_{\mathrm{B}} T e N(0) \sum_{n=0}^{\infty} \left\langle \delta g_{0+} - \delta g^*_{0-} + \delta g^{\mathrm{a}}_{0+} - \delta g^{\mathrm{a}}_{0-} \right\rangle_{\mathrm{F}} . \tag{2.57}$$

On the other hand, the formula for the current density has extra term with $\delta\boldsymbol{A}$ originating from Eq. (2.55). By calculating in the same way as the formula for the charge density, we obtain the formula for the current density:

$$\begin{aligned}\delta\boldsymbol{j} = &-\frac{eN(0)}{2} \int_{-\infty}^{\infty} d\varepsilon \left\langle \boldsymbol{v}_{\mathrm{F}} (\delta g^{\mathrm{R}}_{0+} + \delta g^{\mathrm{R}*}_{0-} + \delta g^{\mathrm{a}\prime}_{0+} - \delta g^{\mathrm{a}\prime}_{0-}) \right\rangle_{\mathrm{F}} \tanh \frac{\varepsilon}{2k_{\mathrm{B}}T} \\ &+ 2e^2 N(0) \frac{i\omega\tau}{1 - i\omega\tau} \langle \boldsymbol{v}_{\mathrm{F}} \boldsymbol{v}_{\mathrm{F}} \rangle_{\mathrm{F}} \delta\boldsymbol{A}.\end{aligned} \tag{2.58}$$

We also can express this current density in terms of $\delta g_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ and $\delta g^{\mathrm{a}}_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ as

$$\begin{aligned}\delta\boldsymbol{j} = &-2\pi i k_{\mathrm{B}} T e N(0) \sum_{n=0}^{\infty} \left\langle \boldsymbol{v}_{\mathrm{F}} (\delta g_{0+} - \delta g^*_{0-} + \delta g^{\mathrm{a}}_{0+} - \delta g^{\mathrm{a}}_{0-}) \right\rangle_{\mathrm{F}} \\ &+ 2e^2 N(0) \frac{i\omega\tau}{1 - i\omega\tau} \langle \boldsymbol{v}_{\mathrm{F}} \boldsymbol{v}_{\mathrm{F}} \rangle_{\mathrm{F}} \delta\boldsymbol{A}.\end{aligned} \tag{2.59}$$

With the similar procedures, the energy gap is given as

$$\delta\Delta = \frac{\mathrm{g}_0}{4i} \int_{-\infty}^{\infty} d\varepsilon \left\langle (\delta f^{\mathrm{R}}_{0+} - \delta \bar{f}^{\mathrm{R}*}_{0-} + \delta f^{\mathrm{a}\prime}_{0+} - \delta f^{\mathrm{a}\prime}_{0-}) \phi^* \right\rangle_{\mathrm{F}} \tanh \frac{\varepsilon}{2k_{\mathrm{B}}T}, \tag{2.60}$$

because $\delta f^{\mathrm{R}}_0 \to 0$ and $\delta f^{\mathrm{a}\prime}_0 \to 0$ for $\varepsilon \to \pm\infty$. We also obtain the energy gap in terms of $\delta f_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) \equiv \delta f^{\mathrm{R}}_{0\pm}(i\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ and $\delta f^{\mathrm{a}}_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) \equiv \delta f^{\mathrm{a}\prime}_{0\pm}(i\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$

$$\delta\Delta = \pi \mathrm{g}_0 k_{\mathrm{B}} T \sum_{n=0}^{\infty} \left\langle \left(\delta f_{0+} + \delta \bar{f}^*_{0-} + \delta f^{\mathrm{a}}_{0+} - \delta f^{\mathrm{a}}_{0-} \right) \phi^* \right\rangle_{\mathrm{F}} . \tag{2.61}$$

Note that the barred functions with subscripts $\pm$ are defined generally by $\bar{g}_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}) = g^*_{0\mp}(\varepsilon_n, -\boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})$ and $\delta\bar{g}_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) = \delta g^*_{0\mp}(\varepsilon_n, -\boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, -\omega)$.

Taking the normal-state limit $\Delta \to 0$ reduces these quantities into

$$\delta\rho_0 = 0, \tag{2.62a}$$

$$\delta\boldsymbol{j} = 2e^2 N(0) \frac{i\omega\tau}{1 - i\omega\tau} \langle \boldsymbol{v}_{\mathrm{F}} \boldsymbol{v}_{\mathrm{F}} \rangle_{\mathrm{F}} \delta\boldsymbol{A}^{\mathrm{ex}}, \tag{2.62b}$$

$$\delta\Delta = 0. \tag{2.62c}$$

Thus, the normal-state conductivity $\underline{\sigma}_{\mathrm{O}}$ is obtained as

$$\delta\boldsymbol{j} = \underline{\sigma}_{\mathrm{O}} \delta\boldsymbol{E}^{\mathrm{ex}}, \quad \underline{\sigma}_{\mathrm{O}} = \frac{2\tau e^2 N(0)}{1 - i\omega\tau} \langle \boldsymbol{v}_{\mathrm{F}} \boldsymbol{v}_{\mathrm{F}} \rangle_{\mathrm{F}} . \tag{2.63}$$

The DC conductivity in normal metals with the spherical Fermi surface is $\sigma_{\mathrm{O}xx} = \sigma_{\mathrm{O}yy} = \sigma_{\mathrm{O}zz} = \tau e^2 n/m$ with $n = (2/3)mN(0)v_{\mathrm{F}}^2$. This result is the same as that from the Drude model.

We can obtain $\delta g_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ and $\delta f_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ as follows. Functions $\delta g_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) \equiv \delta g_0^{\mathrm{R}}(i\varepsilon_n \pm \hbar\omega/2, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ and $\delta f_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) \equiv \delta f_0^{\mathrm{R}}(i\varepsilon_n \pm \hbar\omega/2, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ are given by

$$\delta g_{0+} = -2\frac{\bar{\gamma}\delta\gamma_+ + \gamma_+\delta\bar{\gamma}_+}{(1+\gamma_+\bar{\gamma}_+)(1+\gamma\bar{\gamma})}, \tag{2.64a}$$

$$\delta g_{0-} = -2\frac{\bar{\gamma}_-\delta\gamma_- + \gamma\delta\bar{\gamma}_-}{(1+\gamma\bar{\gamma})(1+\gamma_-\bar{\gamma}_-)}, \tag{2.64b}$$

$$\delta f_{0+} = 2\frac{\delta\gamma_+ - \gamma_+\gamma\delta\bar{\gamma}_+}{(1+\gamma_+\bar{\gamma}_+)(1+\gamma\bar{\gamma})}, \tag{2.64c}$$

$$\delta f_{0-} = 2\frac{\delta\gamma_- - \gamma\gamma_-\delta\bar{\gamma}_-}{(1+\gamma\bar{\gamma})(1+\gamma_-\bar{\gamma}_-)}, \tag{2.64d}$$

with $\gamma_\pm(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}) \equiv \gamma^{\mathrm{R}}(i\varepsilon_n \pm \hbar\omega, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})$, $\bar{\gamma}_\pm(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}) \equiv \bar{\gamma}^{\mathrm{R}}(i\varepsilon_n \pm \hbar\omega, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})$, $\delta\gamma_\pm(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) \equiv \delta\gamma^{\mathrm{R}}(i\varepsilon_n \pm \hbar\omega/2, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ and $\delta\bar{\gamma}_\pm(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) \equiv \delta\bar{\gamma}^{\mathrm{R}}(i\varepsilon_n \pm \hbar\omega/2, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$. It follows from Eq. (2.23) that $\delta\gamma_\pm(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) \equiv \delta\gamma^{\mathrm{R}}(i\varepsilon_n \pm \hbar\omega/2, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ obeys

$$\begin{aligned}
&\hbar\boldsymbol{v}_{\mathrm{F}} \cdot \left(\boldsymbol{\nabla} - i\frac{2e\boldsymbol{A}}{\hbar}\right)\delta\gamma_+ + 2\left[\varepsilon_n - i\frac{\hbar\omega}{2} + \frac{\hbar}{4\tau}(\langle g_{0+}\rangle_{\mathrm{F}} + \langle \bar{g}_0\rangle_{\mathrm{F}})\right]\delta\gamma_+ \\
&\quad - \mathrm{i}e\boldsymbol{v}_{\mathrm{F}} \cdot \delta\boldsymbol{A}(\gamma_+ + \gamma) + \frac{\hbar}{2\tau}(\langle \delta g_{0+}\rangle_{\mathrm{F}}\gamma + \langle \delta\bar{g}_{0+}\rangle_{\mathrm{F}}\gamma_+) \\
&\quad + \left[\left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_{0+}\rangle_{\mathrm{F}}\right)\gamma_+ + \left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_0\rangle_{\mathrm{F}}\right)\gamma\right]\delta\gamma_+ \\
&\quad + \left(\delta\bar{\Delta}\phi^* + \frac{\hbar}{2\tau}\langle \delta\bar{f}_{0+}\rangle_{\mathrm{F}}\right)\gamma_+\gamma - \left(\delta\Delta\phi + \frac{\hbar}{2\tau}\langle \delta f_{0+}\rangle_{\mathrm{F}}\right) = 0,
\end{aligned} \tag{2.65a}$$

$$\begin{aligned}
&\hbar\boldsymbol{v}_{\mathrm{F}} \cdot \left(\boldsymbol{\nabla} - i\frac{2e\boldsymbol{A}}{\hbar}\right)\delta\gamma_- + 2\left[\varepsilon_n + i\frac{\hbar\omega}{2} + \frac{\hbar}{4\tau}(\langle g_0\rangle_{\mathrm{F}} + \langle \bar{g}_{0-}\rangle_{\mathrm{F}})\right]\delta\gamma_- \\
&\quad - \mathrm{i}e\boldsymbol{v}_{\mathrm{F}} \cdot \delta\boldsymbol{A}(\gamma + \gamma_-) + \frac{\hbar}{2\tau}(\langle \delta g_{0-}\rangle_{\mathrm{F}}\gamma_- + \langle \delta\bar{g}_{0-}\rangle_{\mathrm{F}}\gamma) \\
&\quad + \left[\left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_0\rangle_{\mathrm{F}}\right)\gamma + \left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_{0-}\rangle_{\mathrm{F}}\right)\gamma_-\right]\delta\gamma_- \\
&\quad + \left(\delta\bar{\Delta}\phi^* + \frac{\hbar}{2\tau}\langle \delta\bar{f}_{0-}\rangle_{\mathrm{F}}\right)\gamma\gamma_- - \left(\delta\Delta\phi + \frac{\hbar}{2\tau}\langle \delta f_{0-}\rangle_{\mathrm{F}}\right) = 0,
\end{aligned} \tag{2.65b}$$

where $g_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}) \equiv g_0^{\mathrm{R}}(i\varepsilon_n \pm \hbar\omega, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})$ and $f_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}) \equiv f_0^{\mathrm{R}}(i\varepsilon_n \pm \hbar\omega, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})$ are defined by

$$g_{0\pm} = \frac{1 - \gamma_\pm\bar{\gamma}_\pm}{1 + \gamma_\pm\bar{\gamma}_\pm}, \tag{2.66a}$$

$$f_{0\pm} = \frac{2\gamma_\pm}{1 + \gamma_\pm\bar{\gamma}_\pm}. \tag{2.66b}$$

Equations for $\gamma_\pm$ are obtained from Eq. (2.19) as

$$\hbar \boldsymbol{v}_\mathrm{F} \cdot \left(\boldsymbol{\nabla} - i\frac{2e\mathbf{A}}{\hbar} \right) \gamma_\pm + 2\left[\varepsilon_n \mp i\hbar\omega + \frac{\hbar}{4\tau}(\langle g_{0\pm}\rangle_\mathrm{F} + \langle \bar{g}_{0\pm}\rangle_\mathrm{F}) \right] \gamma_\pm$$
$$+ \left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_\pm\rangle_\mathrm{F} \right) \gamma_\pm^2 - \left(\Delta\phi + \frac{\hbar}{2\tau}\langle f_\pm\rangle_\mathrm{F} \right) = 0. \tag{2.67}$$

We can also obtain $\delta g^\mathrm{a}_{0\pm}(\varepsilon_n, \boldsymbol{p}_\mathrm{F}, \boldsymbol{r}, \omega)$ and $\delta f^\mathrm{a}_{0\pm}(\varepsilon_n, \boldsymbol{p}_\mathrm{F}, \boldsymbol{r}, \omega)$ as follows. Noting Eq. (2.28), we can express Eq. (2.50) as

$$\delta g_0^{\mathrm{a}\prime} = 2\frac{\delta x^{\mathrm{a}\prime} - \gamma_+^\mathrm{R}\bar{\gamma}_-^\mathrm{A}\delta\bar{x}^{\mathrm{a}\prime}}{(1+\gamma_+^\mathrm{R}\bar{\gamma}_+^\mathrm{R})(1+\gamma_-^\mathrm{A}\bar{\gamma}_-^\mathrm{A})}, \tag{2.68a}$$

$$\delta f_0^{\mathrm{a}\prime} = 2\frac{\gamma_-^\mathrm{A}\delta x^{\mathrm{a}\prime} + \gamma_+^\mathrm{R}\delta\bar{x}^{\mathrm{a}\prime}}{(1+\gamma_+^\mathrm{R}\bar{\gamma}_+^\mathrm{R})(1+\gamma_-^\mathrm{A}\bar{\gamma}_-^\mathrm{A})}, \tag{2.68b}$$

where $\delta x^{\mathrm{a}\prime}$ is defined as

$$\delta x^{\mathrm{a}\prime} \equiv \delta x^\mathrm{a} \left(\tanh\frac{\varepsilon_-}{2k_\mathrm{B}T} - \tanh\frac{\varepsilon_+}{2k_\mathrm{B}T} \right)^{-1}. \tag{2.69}$$

Equation for $\delta x^{\mathrm{a}\prime}$ is obtained from Eq. (2.30) as

$$\hbar\boldsymbol{v}_\mathrm{F}\cdot\boldsymbol{\nabla}\delta x^{\mathrm{a}\prime} - i\hbar\omega\delta x^{\mathrm{a}\prime} + \frac{\hbar}{2\tau}(\langle g_{0+}^\mathrm{R}\rangle_\mathrm{F} - \langle g_{0-}^\mathrm{A}\rangle_\mathrm{F})\delta x^{\mathrm{a}\prime}$$
$$- \frac{\hbar}{2\tau}(\langle \delta g_0^{\mathrm{a}\prime}\rangle_\mathrm{F} - \langle \delta\bar{g}_0^{\mathrm{a}\prime}\rangle_\mathrm{F}\gamma_+^\mathrm{R}\bar{\gamma}_-^\mathrm{A} + \langle \delta f_0^{\mathrm{a}\prime}\rangle_\mathrm{F}\bar{\gamma}_-^\mathrm{A} + \langle \delta\bar{f}_0^{\mathrm{a}\prime}\rangle_\mathrm{F}\gamma_+^\mathrm{R})$$
$$+ \left[\left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_{0+}^\mathrm{R}\rangle_\mathrm{F} \right)\gamma_+^\mathrm{R} - \left(\Delta\phi + \frac{\hbar}{2\tau}\langle f_{0-}^\mathrm{A}\rangle_\mathrm{F} \right)\bar{\gamma}_-^\mathrm{A} \right]\delta x^{\mathrm{a}\prime}$$
$$+ \delta\bar{\Delta}\phi^*\gamma_+^\mathrm{R} + \delta\Delta\phi\bar{\gamma}_-^\mathrm{A} - ie\boldsymbol{v}_\mathrm{F}\cdot\delta\boldsymbol{A} = 0. \tag{2.70}$$

Using $g_0^\mathrm{A} = -g_0^{\mathrm{R}*}$, $f_0^\mathrm{A} = \bar{f}_0^{\mathrm{R}*}$ and $\gamma^\mathrm{A} = -\bar{\gamma}^{\mathrm{R}*}$, we rewrite Eq. (2.68) as

$$\delta g_0^{\mathrm{a}\prime} = 2\frac{\delta x^{\mathrm{a}\prime} + \gamma_+^\mathrm{R}\gamma_-^{\mathrm{R}*}\delta\bar{x}^{\mathrm{a}\prime}}{(1+\gamma_+^\mathrm{R}\bar{\gamma}_+^\mathrm{R})(1+\gamma_-^{\mathrm{R}*}\bar{\gamma}_-^{\mathrm{R}*})}, \tag{2.71a}$$

$$\delta f_0^{\mathrm{a}\prime} = 2\frac{-\bar{\gamma}_-^{\mathrm{R}*}\delta x^{\mathrm{a}\prime} + \gamma_+^\mathrm{R}\delta\bar{x}^{\mathrm{a}\prime}}{(1+\gamma_+^\mathrm{R}\bar{\gamma}_+^\mathrm{R})(1+\gamma_-^{\mathrm{R}*}\bar{\gamma}_-^{\mathrm{R}*})}, \tag{2.71b}$$

and also Eq. (2.70) for $\delta x^{\mathrm{a}\prime}$ as

$$\hbar\boldsymbol{v}_\mathrm{F}\cdot\boldsymbol{\nabla}\delta x^{\mathrm{a}\prime} - i\hbar\omega\delta x^{\mathrm{a}\prime} + \frac{\hbar}{2\tau}(\langle g_{0+}^\mathrm{R}\rangle_\mathrm{F} + \langle g_{0-}^{\mathrm{R}*}\rangle_\mathrm{F})\delta x^{\mathrm{a}\prime}$$
$$- \frac{\hbar}{2\tau}(\langle \delta g_0^{\mathrm{a}\prime}\rangle_\mathrm{F} + \langle \delta\bar{g}_0^{\mathrm{a}\prime}\rangle_\mathrm{F}\gamma_+^\mathrm{R}\gamma_-^{\mathrm{R}*} - \langle \delta f_0^{\mathrm{a}\prime}\rangle_\mathrm{F}\gamma_-^{\mathrm{R}*} + \langle \delta\bar{f}_0^{\mathrm{a}\prime}\rangle_\mathrm{F}\gamma_+^\mathrm{R})$$
$$+ \left[\left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_{0+}^\mathrm{R}\rangle_\mathrm{F} \right)\gamma_+^\mathrm{R} + \left(\Delta\phi + \frac{\hbar}{2\tau}\langle \bar{f}_{0-}^{\mathrm{R}\,*}\rangle_\mathrm{F} \right)\gamma_-^{\mathrm{R}*} \right]\delta x^{\mathrm{a}\prime}$$
$$+ \delta\bar{\Delta}\phi^*\gamma_+^\mathrm{R} - \delta\Delta\phi\gamma_-^{\mathrm{R}*} - ie\boldsymbol{v}_\mathrm{F}\cdot\delta\boldsymbol{A} = 0. \tag{2.72}$$

Accordingly, $\delta g^{\mathrm{a}}_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ and $\delta f^{\mathrm{a}}_{0\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ are given by

$$\delta g^{\mathrm{a}}_{0+} = 2\frac{\delta x^{\mathrm{a}}_{+} + \gamma_+ \gamma^* \delta \bar{x}^{\mathrm{a}}_{+}}{(1+\gamma_+\bar{\gamma}_+)(1+\gamma^*\bar{\gamma}^*)}, \tag{2.73a}$$

$$\delta g^{\mathrm{a}}_{0-} = 2\frac{\delta x^{\mathrm{a}}_{-} + \gamma \gamma^*_- \delta \bar{x}^{\mathrm{a}}_{-}}{(1+\gamma\bar{\gamma})(1+\gamma^*_-\bar{\gamma}^*_-)}, \tag{2.73b}$$

$$\delta f^{\mathrm{a}}_{0+} = 2\frac{-\bar{\gamma}^* \delta x^{\mathrm{a}}_{+} + \gamma_+ \delta \bar{x}^{\mathrm{a}}_{+}}{(1+\gamma_+\bar{\gamma}_+)(1+\gamma^*\bar{\gamma}^*)}, \tag{2.73c}$$

$$\delta f^{\mathrm{a}}_{0-} = 2\frac{-\bar{\gamma}^*_- \delta x^{\mathrm{a}}_{-} + \gamma \delta \bar{x}^{\mathrm{a}}_{-}}{(1+\gamma\bar{\gamma})(1+\gamma^*_-\bar{\gamma}^*_-)}, \tag{2.73d}$$

with $\delta x^{\mathrm{a}}_{\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) \equiv \delta x^{\mathrm{a}\prime}(i\varepsilon_n \pm \hbar\omega/2, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ and $\delta \bar{x}^{\mathrm{a}}_{\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) \equiv \delta \bar{x}^{\mathrm{a}\prime}(i\varepsilon_n \pm \hbar\omega/2, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$. Equations for $\delta x^{\mathrm{a}}_{\pm}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega) \equiv \delta x^{\mathrm{a}\prime}(i\varepsilon_n \pm \hbar\omega/2, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}, \omega)$ are given as

$$\begin{aligned}
&\hbar \boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\nabla} \delta x^{\mathrm{a}}_{+} - i\hbar\omega \delta x^{\mathrm{a}}_{+} + \frac{\hbar}{2\tau}(\langle g_{0+}\rangle_{\mathrm{F}} + \langle g^*_0\rangle_{\mathrm{F}})\delta x^{\mathrm{a}}_{+} \\
&\quad - \frac{\hbar}{2\tau}(\langle \delta g^{\mathrm{a}}_{0+}\rangle_{\mathrm{F}} + \langle \delta \bar{g}^{\mathrm{a}}_{0+}\rangle_{\mathrm{F}}\gamma_+\gamma^* - \langle \delta f^{\mathrm{a}}_{0+}\rangle_{\mathrm{F}}\gamma^* + \langle \delta \bar{f}^{\mathrm{a}}_{0+}\rangle_{\mathrm{F}}\gamma_+) \\
&\quad + \left[\left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_{0+}\rangle_{\mathrm{F}}\right)\gamma_+ + \left(\Delta\phi + \frac{\hbar}{2\tau}\langle f^*_0\rangle_{\mathrm{F}}\right)\gamma^*\right]\delta x^{\mathrm{a}}_{+} \\
&\quad + \delta\bar{\Delta}\phi^*\gamma_+ - \delta\Delta\phi\gamma^* - ie\boldsymbol{v}_{\mathrm{F}} \cdot \delta\boldsymbol{A} = 0,
\end{aligned} \tag{2.74a}$$

$$\begin{aligned}
&\hbar \boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\nabla} \delta x^{\mathrm{a}}_{-} - i\hbar\omega \delta x^{\mathrm{a}}_{-} + \frac{\hbar}{2\tau}(\langle g_{0}\rangle_{\mathrm{F}} + \langle g^*_{0-}\rangle_{\mathrm{F}})\delta x^{\mathrm{a}}_{-} \\
&\quad - \frac{\hbar}{2\tau}(\langle \delta g^{\mathrm{a}}_{0-}\rangle_{\mathrm{F}} + \langle \delta \bar{g}^{\mathrm{a}}_{0-}\rangle_{\mathrm{F}}\gamma\gamma^*_- - \langle \delta f^{\mathrm{a}}_{0-}\rangle_{\mathrm{F}}\gamma^*_- + \langle \delta \bar{f}^{\mathrm{a}}_{0-}\rangle_{\mathrm{F}}\gamma) \\
&\quad + \left[\left(\Delta^*\phi^* + \frac{\hbar}{2\tau}\langle \bar{f}_{0}\rangle_{\mathrm{F}}\right)\gamma + \left(\Delta\phi + \frac{\hbar}{2\tau}\langle f^*_{0-}\rangle_{\mathrm{F}}\right)\gamma^*_-\right]\delta x^{\mathrm{a}}_{-} \\
&\quad + \delta\bar{\Delta}\phi^*\gamma - \delta\Delta\phi\gamma^*_- - ie\boldsymbol{v}_{\mathrm{F}} \cdot \delta\boldsymbol{A} = 0.
\end{aligned} \tag{2.74b}$$

Finally, we focus on the Hall electric field both in equilibrium and in the flux-flow state. To solve Eqs. (2.42a) and (2.42b) efficiently, we derive equations for $\int_{-\infty}^{\infty} d\varepsilon g^{\mathrm{K}\prime}_1$ in the Matsubara formalism as follows. First, let us perform integration with respect to ε for Eqs. (2.35) and (2.39) and adopt the same procedures as those for calculating Eqs. (2.56), (2.58) and (2.60). Then we obtain

$$\boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\nabla} \int_{-\infty}^{\infty} d\varepsilon g^{\mathrm{K}\prime}_1 = -e(\boldsymbol{v}_{\mathrm{F}} \times \boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \int_{-\infty}^{\infty} d\varepsilon (g^{\mathrm{R}}_0 + g^{\mathrm{R}*}_0) \tanh\frac{\varepsilon}{2k_{\mathrm{B}}T}, \tag{2.75a}$$

$$\begin{aligned}
\boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\nabla} \int_{-\infty}^{\infty} d\varepsilon \delta g^{\mathrm{K}\prime}_1 &= 4\frac{i\omega\tau}{1-i\omega\tau} e^2 (\boldsymbol{v}_{\mathrm{F}} \times \boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \boldsymbol{v}_{\mathrm{F}} \cdot \delta\boldsymbol{A} \\
&\quad - e(\boldsymbol{v}_{\mathrm{F}} \times \boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \int_{-\infty}^{\infty} d\varepsilon (\delta g^{\mathrm{R}}_{0+} + \delta g^{\mathrm{R}*}_{0-} + \delta g^{\mathrm{a}\prime}_{0+} - \delta g^{\mathrm{a}\prime}_{0-}) \tanh\frac{\varepsilon}{2k_{\mathrm{B}}T} \\
&\quad - e(\boldsymbol{v}_{\mathrm{F}} \times \delta\boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \int_{-\infty}^{\infty} d\varepsilon (g^{\mathrm{R}}_0 + g^{\mathrm{R}*}_0) \tanh\frac{\varepsilon}{2k_{\mathrm{B}}T}.
\end{aligned} \tag{2.75b}$$

Closing the integration contour in the upper complex-ε plane and using the residue theorem, we can rewrite Eqs. (2.75a) and (2.75b) as

$$\boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\nabla} \int_{-\infty}^{\infty} d\varepsilon g_1^{\mathrm{K}\prime} = -4\pi i k_{\mathrm{B}} T e(\boldsymbol{v}_{\mathrm{F}} \times \boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \sum_{n=0}^{\infty} (g_0 - g_0^*), \tag{2.76a}$$

$$\begin{aligned} \boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\nabla} \int_{-\infty}^{\infty} d\varepsilon \delta g_1^{\mathrm{K}\prime} &= 4\frac{i\omega\tau}{1-i\omega\tau} e^2 (\boldsymbol{v}_{\mathrm{F}} \times \boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \boldsymbol{v}_{\mathrm{F}} \cdot \delta \boldsymbol{A} \\ &- 4\pi i k_{\mathrm{B}} T e(\boldsymbol{v}_{\mathrm{F}} \times \boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \sum_{n=0}^{\infty} (\delta g_{0+} - \delta g_{0-}^* + \delta g_{0+}^{\mathrm{a}} - \delta g_{0-}^{\mathrm{a}}) \\ &- 4\pi i k_{\mathrm{B}} T e(\boldsymbol{v}_{\mathrm{F}} \times \delta \boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \sum_{n=0}^{\infty} (g_0 - g_0^*). \end{aligned} \tag{2.76b}$$

Solving Eqs. (2.42a), (2.42b), (2.76a) and (2.76b), we can obtain the Hall electric field in equilibrium and in the flux-flow state.

Taking the nomal-state limit $\Delta \to 0$, equation (2.76b) reduces to

$$\boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\nabla} \int_{-\infty}^{\infty} d\varepsilon \delta g_1^{\mathrm{K}\prime} = 4\frac{i\omega\tau}{1-i\omega\tau} e^2 (\boldsymbol{v}_{\mathrm{F}} \times \boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \boldsymbol{v}_{\mathrm{F}} \cdot \delta \boldsymbol{A}^{\mathrm{ex}}, \tag{2.77}$$

which is clearly satisfied by the solution of

$$\boldsymbol{\nabla} \int_{-\infty}^{\infty} d\varepsilon \delta g_1^{\mathrm{K}\prime} = 4\frac{i\omega\tau}{1-i\omega\tau} e^2 \boldsymbol{B} \times \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \boldsymbol{v}_{\mathrm{F}} \cdot \delta \boldsymbol{A}^{\mathrm{ex}}. \tag{2.78}$$

Let us substitute Eq. (2.42b) into Eq. (2.78) and use Eq. (2.63). We thereby obtain

$$\delta \boldsymbol{E}_1 = \boldsymbol{B} \times \underline{R}_{\mathrm{H}}^{(n)} \delta \boldsymbol{j}, \tag{2.79}$$

where $\underline{R}_{\mathrm{H}}^{(n)}$ denotes the normal-state Hall coefficient

$$\underline{R}_{\mathrm{H}}^{(n)} = \frac{1}{2e^2 N(0)} \left\langle \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \boldsymbol{v}_{\mathrm{F}} \right\rangle_{\mathrm{F}} \langle \boldsymbol{v}_{\mathrm{F}} \boldsymbol{v}_{\mathrm{F}} \rangle_{\mathrm{F}}^{-1}. \tag{2.80}$$

The normal-state Hall coefficient with the spherical Fermi surface is $R_{\mathrm{H}xx}^{(n)} = R_{\mathrm{H}yy}^{(n)} = R_{\mathrm{H}zz}^{(n)} = 1/ne$ with n denoting the electron density.

2.6 London equation and equilibrium Hall coefficient

We consider clean superconductors in the Meissner state. Let us neglect the impurity self-energy terms by $\hat{\sigma}_{\mathrm{imp}} \to \hat{0}$, and assume that the spatial variation of the pair potential lies only in its phase as $\Delta(\mathbf{r}) = |\Delta| \, \mathrm{e}^{i\varphi(\boldsymbol{r})}$. We start from the $(1,2)$ element of the standard Eilenberger equations (2.16a) with $\varepsilon \to i\varepsilon_n$ given by

$$2\varepsilon_n f_0 + \hbar \boldsymbol{v}_{\mathrm{F}} \cdot \left(\boldsymbol{\nabla} - i\frac{2e\boldsymbol{A}}{\hbar} \right) f_0 - \Delta\phi \bar{g}_0 - \Delta\phi g_0 = 0, \tag{2.81}$$

with the normalization condition $g_0 = (1 - f_0 \bar{f}_0)^{1/2}$ for $\varepsilon_n > 0$. Regarding the gradient term $\boldsymbol{\nabla} - i2e\boldsymbol{A}/\hbar$ in (2.81) as a perturbation [10, 7], we obtain the first-order equation

$$2\varepsilon_n f_0^{(1)} + \hbar \boldsymbol{v}_\mathrm{F} \cdot \left(\boldsymbol{\nabla} - i\frac{2e\boldsymbol{A}}{\hbar} \right) f_0^{(0)} = 2|\Delta|\phi e^{i\varphi} g_0^{(1)}, \tag{2.82}$$

and $g_0^{(1)}$ for $\varepsilon_n > 0$ is obtained from the normalization condition as

$$g_0^{(1)} = -\frac{f_0^{(0)} \bar{f}_0^{(1)} + f_0^{(1)} \bar{f}_0^{(0)}}{2g_0^{(0)}}, \tag{2.83}$$

where $g_0^{(0)}$ and $f_0^{(0)}$ are the homogeneous solutions given by

$$g_0^{(0)} = \frac{\varepsilon_n}{\sqrt{\varepsilon_n^2 + |\Delta|^2|\phi|^2}}, \tag{2.84a}$$

$$f_0^{(0)} = \frac{|\Delta|\phi e^{i\varphi}}{\sqrt{\varepsilon_n^2 + |\Delta|^2|\phi|^2}}. \tag{2.84b}$$

Then we obtain equations for $f_0^{(0)}$ and $\bar{f}_0^{(0)}$ as

$$2\varepsilon_n f_0^{(1)} + \hbar \boldsymbol{v}_\mathrm{F} \cdot \left(\boldsymbol{\nabla} - i\frac{2e\boldsymbol{A}}{\hbar} \right) f_0^{(0)} = -\frac{|\Delta|^2\phi^2 e^{2i\varphi} \bar{f}_0^{(1)} + |\Delta|^2|\phi|^2 f_0^{(1)}}{2\varepsilon_n}, \tag{2.85a}$$

$$2\varepsilon_n \bar{f}_0^{(1)} - \hbar \boldsymbol{v}_\mathrm{F} \cdot \left(\boldsymbol{\nabla} + i\frac{2e\boldsymbol{A}}{\hbar} \right) \bar{f}_0^{(0)} = -\frac{|\Delta|^2|\phi|^2 \bar{f}_0^{(1)} + |\Delta|^2\phi^{*2} e^{-2i\varphi} f_0^{(1)}}{2\varepsilon_n}. \tag{2.85b}$$

The above equations can be solved easily as

$$f_0^{(1)} = -i\frac{\varepsilon_n|\Delta|\phi e^{i\varphi}}{2(\varepsilon_n^2 + |\Delta|^2|\phi|^2)^{3/2}} \hbar \boldsymbol{v}_\mathrm{F} \cdot \left(\boldsymbol{\nabla}\varphi - \frac{2e\boldsymbol{A}}{\hbar} \right), \tag{2.86a}$$

$$\bar{f}_0^{(1)} = -i\frac{\varepsilon_n|\Delta|\phi^* e^{-i\varphi}}{2(\varepsilon_n^2 + |\Delta|^2|\phi|^2)^{3/2}} \hbar \boldsymbol{v}_\mathrm{F} \cdot \left(\boldsymbol{\nabla}\varphi - \frac{2e\boldsymbol{A}}{\hbar} \right). \tag{2.86b}$$

Substituting Eqs. (2.86a) and (2.86b) into Eq. (2.83), we obtain

$$g_0^{(1)} = i\frac{|\Delta|^2|\phi|^2}{2(\varepsilon_n^2 + |\Delta|^2|\phi|^2)^{3/2}} \hbar \boldsymbol{v}_\mathrm{F} \cdot \left(\boldsymbol{\nabla}\varphi - \frac{2e\boldsymbol{A}}{\hbar} \right). \tag{2.87}$$

To obtain the gap equation, let us approximate $f \approx f_0^{(0)}$, and substitute Eq. (2.84b) into Eq. (2.45b). We then obtain

$$1 = 2\pi \mathrm{g}_0 k_\mathrm{B} T \sum_{n=0}^{\infty} \left\langle \frac{|\phi|^2}{\sqrt{\varepsilon_n^2 + |\Delta|^2|\phi|^2}} \right\rangle_\mathrm{F}. \tag{2.88}$$

The value of $|\Delta|$ in the Meissner state is determined by solving the above equation.

To obtain the London equation, let us approximate $g_0 \approx g_0^{(0)} + g_0^{(1)}$, substitute Eqs. (2.84a) and (2.87) into Eq. (2.45a). We then obtain

$$\boldsymbol{j} = -2e^2 N(0) \langle \boldsymbol{v}_\mathrm{F}(1 - Y)\mathbf{v}_\mathrm{F} \rangle_\mathrm{F} \left(\boldsymbol{A} - \frac{\hbar}{2e}\boldsymbol{\nabla}\varphi \right), \tag{2.89}$$

where Y denotes the Yosida function [10, 7, 47] defined by

$$Y \equiv 1 - 2\pi k_{\rm B}T \sum_{n=0}^{\infty} \frac{|\Delta|^2|\phi|^2}{(\varepsilon_n^2 + |\Delta|^2|\phi|^2)^{3/2}}. \tag{2.90}$$

Substituting Eq. (2.89) into Ampère's law (2.8), we then obtain the equation for the vector potential as

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{A} = -2e^2\mu_0 N(0)\langle \boldsymbol{v}_{\rm F}(1-Y)\mathbf{v}_{\rm F}\rangle_{\rm F}\left(\boldsymbol{A} - \frac{\hbar}{2e}\boldsymbol{\nabla}\varphi\right). \tag{2.91}$$

We also carry out the following procedures to obtain the London equation: (i) Use $\langle(1-Y)v_{{\rm F}x}v_{{\rm F}y}\rangle_{\rm F} = 0$ and $\langle(1-Y)v_{{\rm F}x}^2\rangle_{\rm F} = \langle(1-Y)v_{{\rm F}y}^2\rangle_{\rm F}$. (ii) Operate with $\boldsymbol{\nabla}\times$ on the above equation. (iii) Use identities $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{B} = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\boldsymbol{B} - \boldsymbol{\nabla}^2\boldsymbol{B}$ and $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\varphi = 0$ together with Gauss's Law $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ for magnetism. We thereby obtain the London equation as

$$\boldsymbol{\nabla}^2\boldsymbol{B} = \frac{1}{\lambda_{\rm L}^2}\boldsymbol{B}, \tag{2.92}$$

where $\lambda_{\rm L}$ is the London penetration depth at finite temperatures $\lambda_{\rm L} \equiv \lambda_0\langle v_{\rm F}\rangle_{\rm F}[2\langle(1-Y)v_{{\rm F}x}^2\rangle_{\rm F}]^{-1/2}$ and λ_0 is the magnetic penetration depth $\lambda_0 \equiv [\mu_0 N(0)e^2\langle v_{\rm F}^2\rangle_{\rm F}]^{-1/2}$.

We now solve Eqs. (2.76a) and (2.42a) for the Meissner state. Equation (2.76a) is clearly satisfied by the solution of

$$\boldsymbol{\nabla}\int_{-\infty}^{\infty} d\varepsilon g_1^{{\rm K}\prime} = -4\pi i k_{\rm B}Te\boldsymbol{B}\times\frac{\partial}{\partial\boldsymbol{p}_{\rm F}}\sum_{n=0}^{\infty}(g_0 - g_0^*). \tag{2.93}$$

Substituting Eq. (2.93) into Eq. (2.42a), we obtain

$$-\lambda_{\rm TF}^2\boldsymbol{\nabla}^2\boldsymbol{E}_1 + \boldsymbol{E}_1 = -\pi i k_{\rm B}T\boldsymbol{B}\times\sum_{n=0}^{\infty}\left\langle\frac{\partial}{\partial\boldsymbol{p}_{\rm F}}(g_0 - g_0^*)\right\rangle_{\rm F}. \tag{2.94}$$

Let us approximate $g_0 \approx g_0^{(0)} + g_0^{(1)}$, and use Eqs. (2.84a) and (2.87). We then obtain

$$-\lambda_{\rm TF}^2\boldsymbol{\nabla}^2\boldsymbol{E}_1 + \boldsymbol{E}_1 = -e\boldsymbol{B}\times\left\langle\frac{\partial}{\partial\boldsymbol{p}_{\rm F}}(1-Y)\boldsymbol{v}_{\rm F}\right\rangle_{\rm F}\left(\boldsymbol{A} - \frac{\hbar}{2e}\boldsymbol{\nabla}\varphi\right). \tag{2.95}$$

Using the London equation (2.89), the above equation is expressible as

$$-\lambda_{\rm TF}^2\boldsymbol{\nabla}^2\boldsymbol{E}_1 + \boldsymbol{E}_1 = \boldsymbol{B}\times\underline{R}_{\rm H}\boldsymbol{j}, \tag{2.96}$$

with the tensor Hall coefficient [10]

$$\underline{R}_{\rm H} \equiv \frac{1}{2eN(0)}\left\langle\frac{\partial}{\partial\boldsymbol{p}_{\rm F}}(1-Y)\boldsymbol{v}_{\rm F}\right\rangle_{\rm F}\langle\boldsymbol{v}_{\rm F}(1-Y)\boldsymbol{v}_{\rm F}\rangle_{\rm F}^{-1}. \tag{2.97}$$

In the next chapter, we use these solutions of the gap and London equations as boundary conditions to obtain the solution of the Eilenberger equations in a superconductor with an isolated vortex for deriving the expression for the vortex-core charge in an s-wave superconductor with an isolated vortex.

3 Vortex-Core Charging

3.1 Numerical examples for vortex-core charging in an s-wave superconductor

We calculate the core charge for an isolated vortex of an s-wave superconductor with a cylindrical Fermi surface and $\boldsymbol{B} \parallel \boldsymbol{z}$ centered at the origin in the (x, y) plane in the clean limit. Our numerical procedure is summarized as follows. We first obtain $(g_0, \Delta, \mathrm{B})$ for the isolated s-wave vortex solving the standard Eilenberger equations self-consistently [36, 7]. Next, substituting the solution of the Eilengerger equations into Eq. (2.76a), we solve Eq. (2.76a) using the following boundary condition: $\int_{-\infty}^{\infty} d\varepsilon g_1^{\mathrm{K}} = 0$ for $r \gg \xi_0$. The resulting solution is used subsequently to calculate the electric field and charge using Eq. (2.42a) and $\rho = \epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}$, respectively.

The parameters of this system are the coherence length ξ_0, magnetic penetration depth λ_0, Thomas-Fermi screening length λ_{TF}, and quasiclassical parameter δ. Our results below were obtained for $\lambda_0 = 5\xi_0$, $\lambda_{\mathrm{TF}} = 0.01\xi_0$, and $\delta = 0.01$.

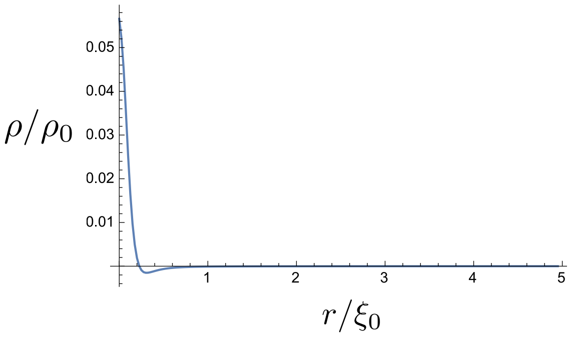

Figure 1: Charge density $\rho(r)$ at $T = 0.3T_{\mathrm{c}}$ in units of $\rho_0 \equiv \epsilon_0 \Delta_0 / |e| \xi_0^2$ over $0 \leq r \leq 5\xi_0$.

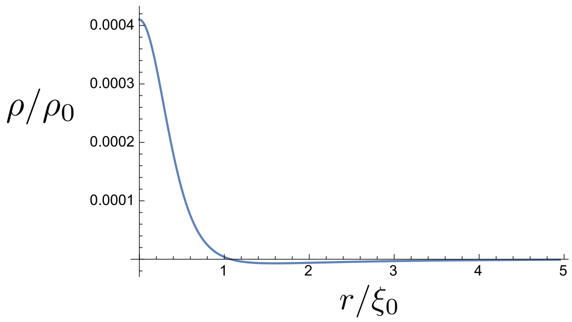

Figure 2: Charge density $\rho(r)$ at $T = 0.7T_{\mathrm{c}}$ in units of $\rho_0 \equiv \epsilon_0 \Delta_0 / |e| \xi_0^2$ over $0 \leq r \leq 5\xi_0$.

Figures 1 and 2 plot the charge density in the core region at $T/T_{\mathrm{c}} = 0.3$ and $T/T_{\mathrm{c}} = 0.7$, respectively, where T_{c} denotes the superconducting transition temperature at zero magnetic field. Compared with the case of $T/T_{\mathrm{c}} = 0.3$, the spatial variation of the charge density at $T/T_{\mathrm{c}} = 0.7$ extends far outside the core. This charge extension is nearly equal to the London penetration depth at finite temperatures λ_{L}. Figure 3 plots the London penetration depth at finite temperatures given by $\lambda_{\mathrm{L}} = \lambda_0 \langle v_{\mathrm{F}} \rangle_{\mathrm{F}} \left[2\langle (1 - Y) v_{\mathrm{F}x}^2 \rangle_{\mathrm{F}} \right]^{-1/2}$ as a function of temperature. Also, the magnitude of core charge is decreased due to the decreasing supercurrent and pair potential as the temperature is increased from $T = 0$.

3.2 Expression for Vortex-Core Charge of extreme type-II materials

We derive an analytic expression for the vortex-core charge due to the Lorentz force in type-II superconductors with an isolated vortex [9]. Using Eq. (2.96) and the charge

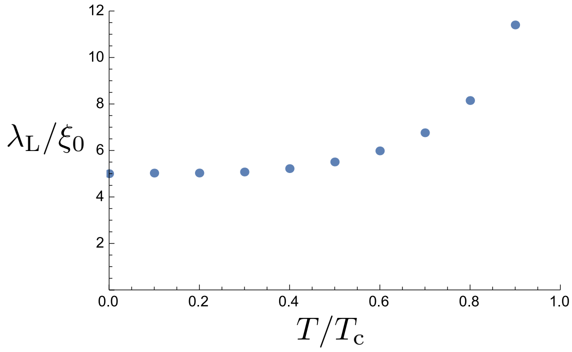

Figure 3: Temperature dependence of the London penetration depth.

neutrality condition, we can calculate the core charge of an isolated vortex analytically for extreme type-II materials such as high-T_c superconductors in the clean limit. We start from the electric field outside the core obtained from Eq. (2.96) as

$$\boldsymbol{E}_1 = \boldsymbol{B} \times \underline{R}_\mathrm{H}\boldsymbol{j}. \tag{3.1}$$

Assuming cylindrical symmetry outside the core, we can express the flux density and supercurrent in the region as [7]

$$B(r) = \frac{\Phi_0}{2\pi\lambda_\mathrm{L}^2} K_0\left(\frac{r}{\lambda_\mathrm{L}}\right), \tag{3.2}$$

$$j(r) = \frac{\Phi_0}{2\pi\lambda_\mathrm{L}^3\mu_0} K_1\left(\frac{r}{\lambda_\mathrm{L}}\right), \tag{3.3}$$

where $K_{0,1}(x)$ are the modified Bessel functions, $\Phi_0 \equiv h/2|e|$ is the magnetic flux quantum and μ_0 is the vacuum permeability. Substituting them into Eq. (3.1), we obtain the electric field along the radial direction as

$$E(r) = -\frac{R_\mathrm{H}\Phi_0^2}{4\pi^2\lambda_\mathrm{L}^5\mu_0} K_0\left(\frac{r}{\lambda_\mathrm{L}}\right) K_1\left(\frac{r}{\lambda_\mathrm{L}}\right), \tag{3.4}$$

where R_H denotes the diagonal element of $\underline{R}_\mathrm{H}$. Integrating the resulting charge density $\rho = \epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}$ over $r_\mathrm{c} \leq r \leq \infty$ with $r_\mathrm{c} \sim \xi_0$, we can estimate the charge accumulated in the outer region per unit length along the flux line. Due to the charge neutrality condition, this charge should be equal in magnitude and opposite in sign to that in $r \leq r_\mathrm{c}$. Approximating $K_0(x) \approx -\ln x$ and $K_1(x) \approx 1/x$ for $x \equiv r_\mathrm{c}/\lambda_\mathrm{L} \ll 1$, we thereby obtain the expression for the vortex-core charge of an isolated vortex within $r \leq r_\mathrm{c}$ per unit length along the flux line as

$$Q_\lambda = -\frac{\epsilon_0 R_\mathrm{H}\Phi_0^2 r_\mathrm{c}}{2\pi\lambda_\mathrm{L}^5\mu_0} K_0\left(\frac{r_\mathrm{c}}{\lambda_\mathrm{L}}\right) K_1\left(\frac{r_\mathrm{c}}{\lambda_\mathrm{L}}\right) \approx \frac{e^2 R_\mathrm{H}}{32\pi\alpha^2\lambda_\mathrm{L}^4} \ln \frac{r_\mathrm{c}}{\lambda_\mathrm{L}}, \tag{3.5}$$

where $\alpha \equiv e^2/4\pi\epsilon_0\hbar c$ is the fine-structure constant with c the light velocity. Equation (3.5) is given in terms of the London penetration depth and the equilibrium Hall coefficient, and implies that the magnitude of the vortex-core charge depends crucially on $\lambda_{\rm L}$. Note also that both the sign and magnitude of Q_λ are strongly affected by the curvature of the Fermi surface and may also exhibit substantial temperature dependence in the presence of gap anisotropy due to the factor $Y = Y(\boldsymbol{p}_{\rm F}, T)$.

For high-$T_{\rm c}$ superconductors with the magnetic field along the c-axis, we substitute $\xi_0 \sim 20$ Å and $\lambda_{\rm L} = 100\xi_0$ into Eq. (3.5), and estimate the vortex-core charge accumulated over the length $\Delta z \sim 5$ Å along the flux line as $|Q| \equiv |Q_\lambda|\Delta z \sim 10^{-5}|e|$. This charge is much smaller than the previous $|Q| \sim 10^{-3}|e|$ [23] and $|Q| \sim 10^{-4}|e|$ [24]. On the other hand, the vortex-core charge due to the Lorentz force has been shown to have a strong magnetic-field dependence with a peak structure and can be enhanced significantly from the value of an isolated vortex as the magnetic field is increased [43, 44].

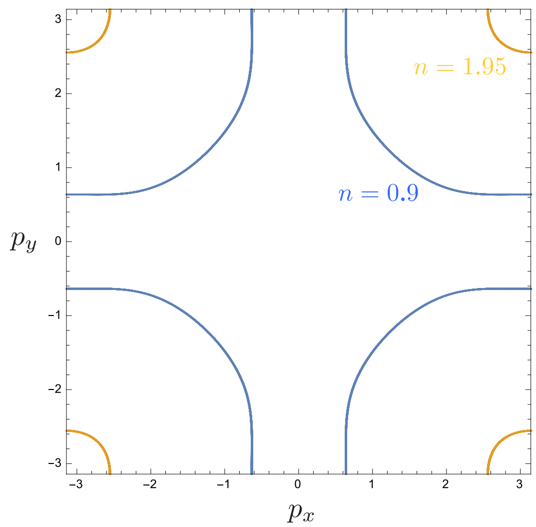

Figure 4: Fermi surfaces at $n = 0.9$ and 1.95 for the single-particle energy given by Eq. (3.6).

3.3 Numerical examples for vortex-core charging in a d-wave superconductor

We calculate the core charge for an isolated vortex of a d-wave superconductor with a anisotropic Fermi surface. To this end, we use the dimensionless single-particle energy of a two-dimensional square lattice appropriate for high-$T_{\rm c}$ superconductors [48, 10]

$$\varepsilon_{\boldsymbol{p}} = -2(\cos p_x + \cos p_y) + 4t_1(\cos p_x \cos p_y - 1) + 2t_2(\cos 2p_x + \cos 2p_y - 2) \tag{3.6}$$

with $t_1 = 1/6$ and $t_2 = -1/5$, which forms a band over $-4 \leq \varepsilon_{\boldsymbol{p}} \leq 4$. Figure 4 shows the Fermi surfaces at $n = 0.9$ and 1.95, where n denotes the average electron filling per site for the single-particle energy given by Eq. (3.6). We also adopt the pair potential given by $\Delta(\boldsymbol{p}_{\rm F}, \boldsymbol{r}) = \Delta(\boldsymbol{r})\phi(\boldsymbol{p}_{\rm F})e^{-i\varphi}$, where $\varphi \equiv \arctan(y/x)$, and $\phi(\boldsymbol{p}_{\rm F})$ is modeled for $n \gtrsim 0.8$ as $\phi(\boldsymbol{p}_{\rm F}) = C\,[(p_{{\rm F}x} - \pi)^2 - (p_{{\rm F}y} - \pi)^2]$ with C denoting the normalization constant determined by $\langle|\phi|^2\rangle_{\rm F} = 1$. The numerical procedure and coordinate system for d-wave pairing is similar to that for s-wave pairing. We chose $\lambda_0 = 100\xi_0$, $\lambda_{\rm TF} = 0.05\xi_0$, and $\delta = 0.05$ as appropriate for high-$T_{\rm c}$ superconductors.

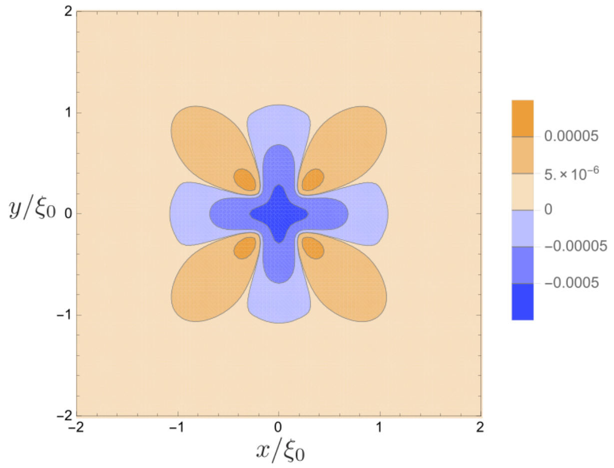

Figure 5: Charge density $\rho(\boldsymbol{r})$ at $T = 0.2T_{\rm c}$ in units of $\rho_0 \equiv \epsilon_0\Delta_0/|e|\xi_0^2$ over $-2\xi_0 \leq x, y \leq 2\xi_0$ at $n = 1.95$ with an isotropic holelike Fermi surface.

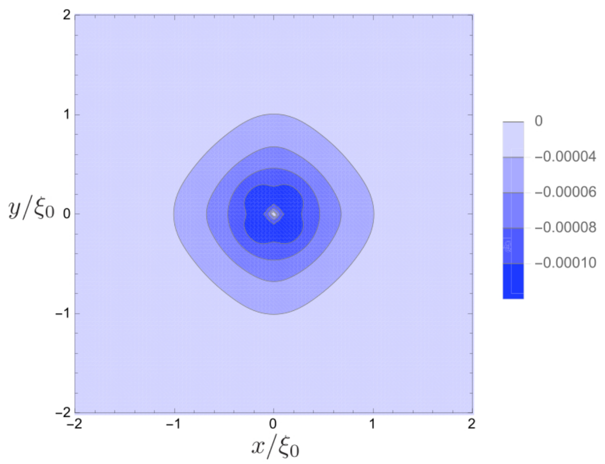

Figure 6: Electric field along the radial direction $E_r(\boldsymbol{r})$ at $T = 0.2T_{\rm c}$ in units of $E_0 \equiv \Delta_0/|e|\xi_0$ over $-2\xi_0 \leq x, y \leq 2\xi_0$ at $n = 1.95$ with an isotropic holelike Fermi surface.

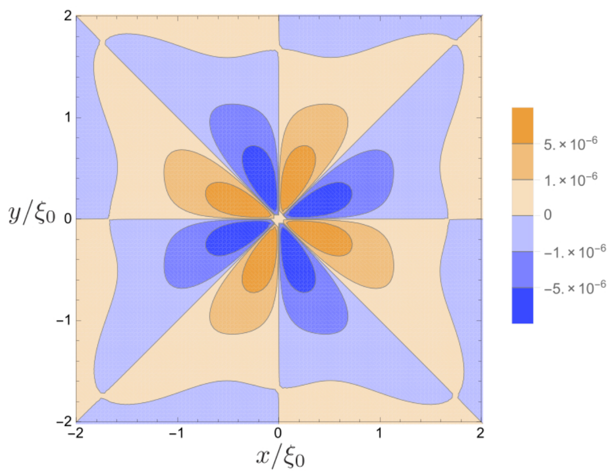

Figure 7: Electric field along the angular direction $E_\varphi(\boldsymbol{r})$ at $T = 0.2T_{\rm c}$ in units of $E_0 \equiv \Delta_0/|e|\xi_0$ over $-2\xi_0 \leq x, y \leq 2\xi_0$ at $n = 1.95$ with an isotropic holelike Fermi surface.

Figure 5 plots the charge density in the core region for $n = 1.95$ with an almost isotropic holelike Fermi surface at $T/T_{\rm c} = 0.2$. We observe that the fourfold symmetry in the core region is due solely to the gap anisotropy, which becomes obscure outside the core region. The sign of the core charge for this holelike Fermi surface is negative, as pointed out previously [23]. Figures 6 and 7 plot the electric field of the radial and angular components in the core region for $n = 1.95$ at $T/T_{\rm c} = 0.2$, respectively. The whole sign of the charge density and electric field is reversed for $n = 0.05$ with the electron-like Fermi surface.

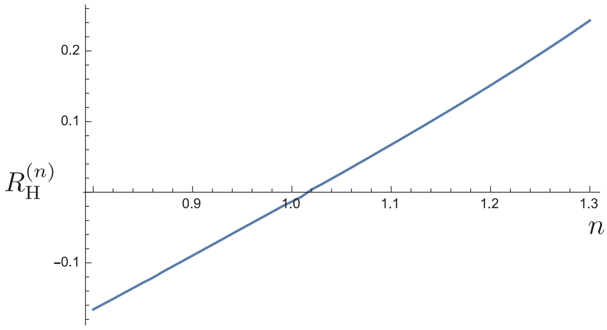

Figure 8: The normal-state Hall coefficient R_{H} given by Eq. (2.80) as a function of the filling n for the single-particle dispersion of Eq. (3.6).

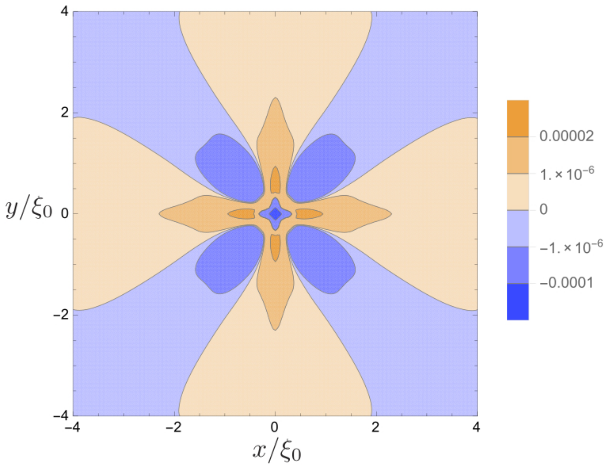

Figure 9: Charge density $\rho(\boldsymbol{r})$ in units of $\rho_0 \equiv \epsilon_0\Delta_0/|e|\xi_0^2$ over $-4\xi_0 \le x, y \le 4\xi_0$ for $n = 0.9$ at $T = 0.2T_{\mathrm{c}}$.

On the other hand, the charge density for a realistic case of $n = 0.9$ exhibits more complicated spatial and temperature dependences. This filling is close to $n_{\mathrm{c}} = 1.03$, where the normal Hall coefficient (2.80) changes its sign as shown in Fig. 8, so that we expect a substantial effect of the Fermi surface anisotropy on the charge distribution according to Eq. (2.97). Figure 9 plots the charge density in the core region at $T/T_{\mathrm{c}} = 0.2$. Here, the sign of charge at the core center is negative, which is reversed in the adjacent region, and the integrated charge over $r \le \xi_0$ and $r \le 2\xi_0$ is found to be positive. Compared with the case of $n = 1.95$, the fourfold symmetry is clearer here and extends far outside the core, which may be attributed to the cooperative effect of the gap and Fermi surface anisotropies. Figures 10 and 11 plot the electric field along the radial and angular directions in the core region for $n = 0.9$ at $T/T_{\mathrm{c}} = 0.2$, respectively. Figure 12 also plots the Hall coefficient of equilibrium supercurrent given by Eq. (2.97) as a function of temperature at $n = 0.9$, and 1.95. The equilibrium Hall coefficient changes its sign due

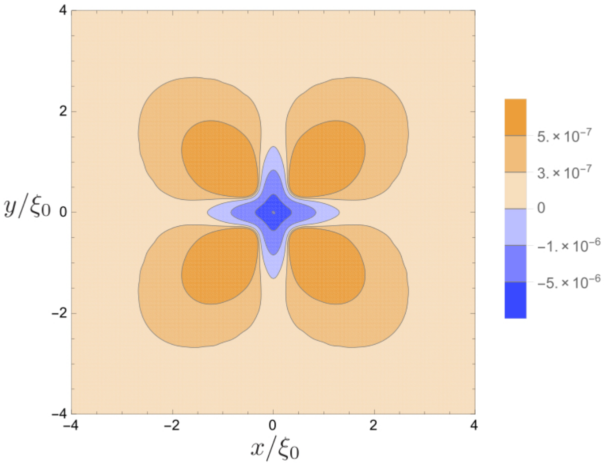

Figure 10: Electric field along the radial direction $E_r(\boldsymbol{r})$ in units of $E_0 \equiv \Delta_0/|e|\xi_0$ over $-4\xi_0 \leq x, y \leq 4\xi_0$ for $n = 0.9$ at $T = 0.2T_c$.

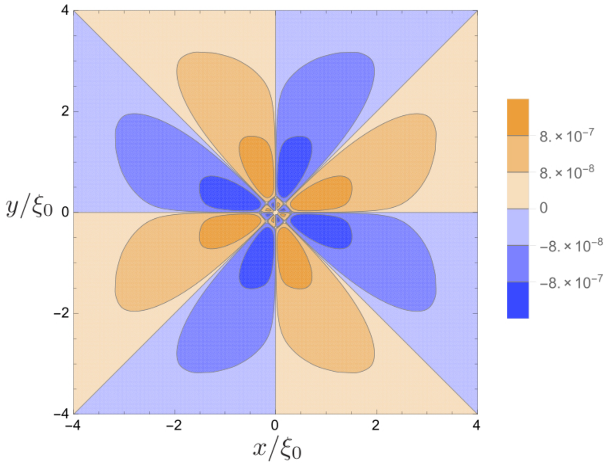

Figure 11: Electric field along the angular direction $E_\varphi(\boldsymbol{r})$ in units of $E_0 \equiv \Delta_0/|e|\xi_0$ over $-4\xi_0 \leq x, y \leq 4\xi_0$ for $n = 0.9$ at $T = 0.2T_c$.

to the variation of the excitation curvature under the growing energy gap as $T \to 0$ in the case of $n = 0.9$. Similarly, the sign change of the Hall electric field between the core region and outside the region may be also caused by the spatial variation in the excitation curvature due to the under the growing energy gap as $r \to \infty$.

Figure 13 plots the temperature dependence of the vortex-core charge Q_λ accumulated within $r \leq 2\xi_0$ for the fillings $0.8 \leq n \leq 1.2$. It shows clearly that both the magnitude and sign of the vortex-core charge change as functions of temperature. Its absolute value decreases as the temperature is raised, which is caused mainly by the increase of the London penetration depth given by $\lambda_L = \lambda_0 \langle v_F \rangle_F \left[2\langle (1-Y) v_{Fx}^2 \rangle_F\right]^{-1/2}$ as $T \to T_c$ as is the case with s-wave pairing. The accumulated charge in the core region can also change its sign, which originates from the sign change of the superconducting Hall coefficient (2.97) due to the variation of the excitation curvature under the growing energy gap as $T \to 0$. This sign change is beyond the scope of the earlier studies based on the density of states [23, 24] and may be regarded as a definite outcome of our microscopic approach. We have confirmed that Eq. (3.5) with $r_c = 2\xi_0$ can reproduce numerical results quantitatively.

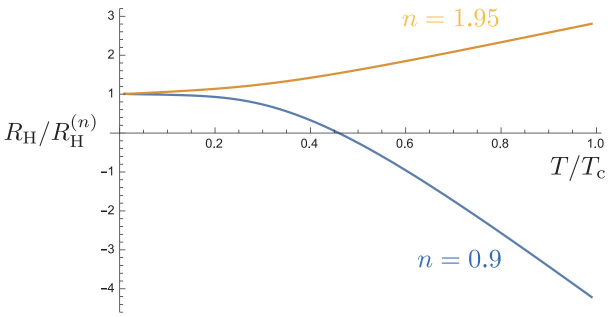

Figure 12: The superconducting Hall coefficient R_{H} given by Eq. (2.97) normalized by the normal-state Hall coefficient $R_{\mathrm{H}}^{(\mathrm{n})}$ as a function of temperature for the fillings $n = 0.9, 1.95$.

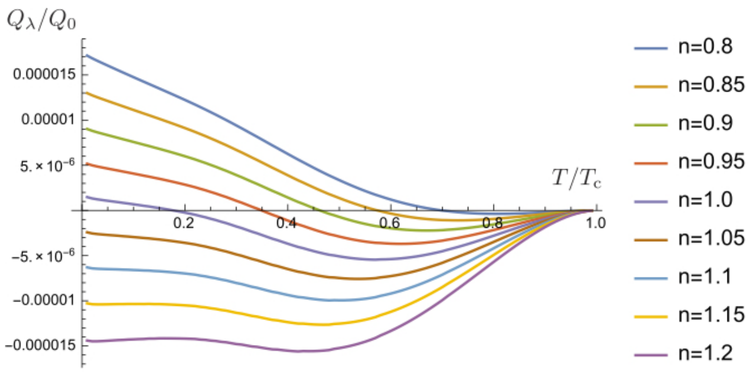

Figure 13: Vortex-core charge Q_λ per unit length along the flux line in units of $Q_0 \equiv \epsilon_0\Delta_0/|e|$ as a function of temperature over $0.8 \le n \le 1.2$.

4 Flux-Flow Hall Effect

4.1 Numerical examples for an s-wave superconductor with an isolated vortex

A microscopic calculation on the flux-flow Hall effect was started by Arahata and Kato [16] for an isolated vortex in an s-wave superconductor. They adopted an approach different from this thesis. First, they sought the solutions to the augmented quasiclassical equations of superconductivity in the form of [39, 40, 16]

$$X(\boldsymbol{r},t) = X^{\mathrm{eq}}(\boldsymbol{r}-\boldsymbol{v}_{\mathrm{v}}t) + \delta X(\boldsymbol{r},t), \tag{4.1}$$

where $\boldsymbol{v}_{\mathrm{v}}$ is the vortex velocity, X^{eq} is the solution in equilibrium, and δX is a term of the first order in $|\boldsymbol{v}_{\mathrm{v}}|$. Next, the notation $a \circ b$ was approximated as

$$a \circ b \approx ab + \frac{\mathrm{i}\hbar}{2}\left(\frac{\partial a}{\partial \varepsilon}\frac{\partial b}{\partial t} - \frac{\partial a}{\partial t}\frac{\partial b}{\partial \varepsilon}\right). \tag{4.2}$$

Accordingly, the time derivative of X^{eq} and δX is given by

$$\frac{\partial X^{\mathrm{eq}}}{\partial t} = -\boldsymbol{v}_{\mathrm{v}} \cdot \boldsymbol{\nabla} X^{\mathrm{eq}}, \quad \frac{\partial \delta X}{\partial t} = 0. \tag{4.3}$$

With these preliminaries, they solved the augmented quasiclassical equations of superconductivity, the gap equation, and Maxwell equations simultaneously in a self-consistent way without the expansion in the quasiclassical parameter.

We here develop an approach different from theirs. Specifically, we calculate linear responses to an external vector potential with frequency ω and take the limit $\omega \to 0$ eventually to study the DC responses we want to know. The details are given as follows. We solve Eqs. (2.65a), (2.65b), (2.74a) and (2.74b) substituting the equilibrium solutions into the source terms and adopting the boundary conditions: $\langle \delta g_{0\pm}\rangle_{\mathrm{F}} \to 0$, $\langle \delta f_{0\pm}\rangle_{\mathrm{F}} \to 0$, $\langle \delta g^{\mathrm{a}}_{0\pm}\rangle_{\mathrm{F}} \to 0$, $\langle \delta f^{\mathrm{a}}_{0\pm}\rangle_{\mathrm{F}} \to 0$, $\delta \boldsymbol{A} \to \delta \boldsymbol{A}^{\mathrm{ex}}$ and $\delta\Delta \to 0$ for $r \gg \xi_0$. Using $\delta g_{0\pm}$, $\delta f_{0\pm}$, $\delta g^{\mathrm{a}}_{0\pm}$ and $\delta f^{\mathrm{a}}_{0\pm}$ thereby obtained, we calculate $\delta\rho_0$ of Eq. (2.57), $\delta \boldsymbol{j}$ of Eq. (2.59), $\delta\Delta$ of Eq. (2.61) and the impurity self-energy. Then, we use the Maxwell equations to obtain $\delta\boldsymbol{A}$ and $\delta\boldsymbol{E}_0$. The procedure should be repeated until numerical convergence in $\delta\Delta$, $\delta\boldsymbol{A}$ and the impurity self-energy is reached. Finally, we can obtain the Hall electric field $\delta\boldsymbol{E}_1$ of Eq. (2.42a), using the solutions of the Eschrig's transport equations and Eq. (2.76a). We have chosen $\lambda_0 = 5\xi_0$, $\lambda_{\mathrm{TF}} = 0.01\xi_0$, $\delta = 0.01$, $\tau = 10\hbar/\Delta_0$, $\omega = 0.01\Delta_0/\hbar$. Also, we have set the external vector potential as $\delta\boldsymbol{A}^{\mathrm{ex}} = -0.01 A_0 \hat{\boldsymbol{x}}$ in units of $A_0 \equiv \hbar/2|e|\xi_0$. We adopt the coordinate system where the transport current flows along the positive x-axis. Observable quantities in the limit $\omega \to 0$ can be obtained from the real parts in the response functions.

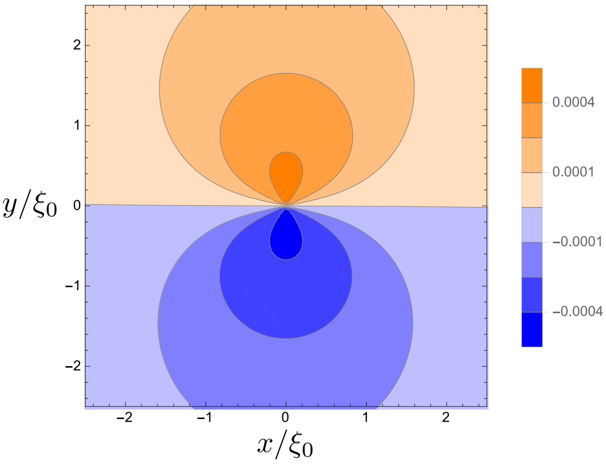

Figure 14: Pair potential $\delta\Delta$ induced by the vortex motion in units of Δ_0 over $-2.5\xi_0 \leq x, y \leq 2.5\xi_0$ at $T/T_c = 0.8$.

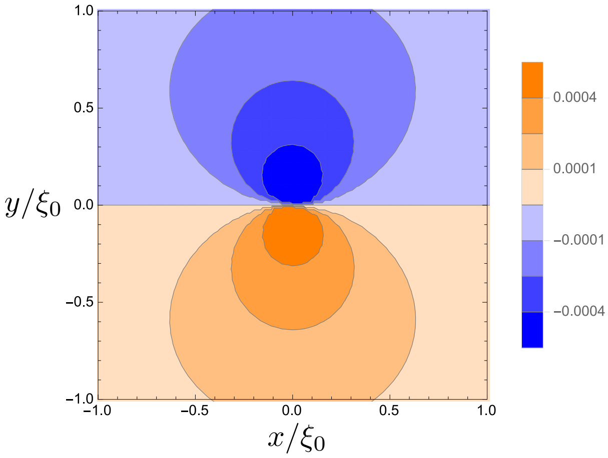

Figure 15: Magnetic field $\delta\boldsymbol{B}$ induced by the vortex motion in units of $B_0 \equiv \hbar/2|e|\xi_0^2$ over $-2.5\xi_0 \leq x, y \leq 2.5\xi_0$ at $T/T_c = 0.8$.

4.1.1 Solution for the standard Eilenberger equations without Hall terms

Figures 14 and 15 plot the pair potential $\delta\Delta$ and magnetic field $\delta\boldsymbol{B}$, respectively, induced by the vortex motion in the core region at $T/T_c = 0.8$. We observe that the isolated vortex moves in the direction of the negative y-axis due to the Magnus force.

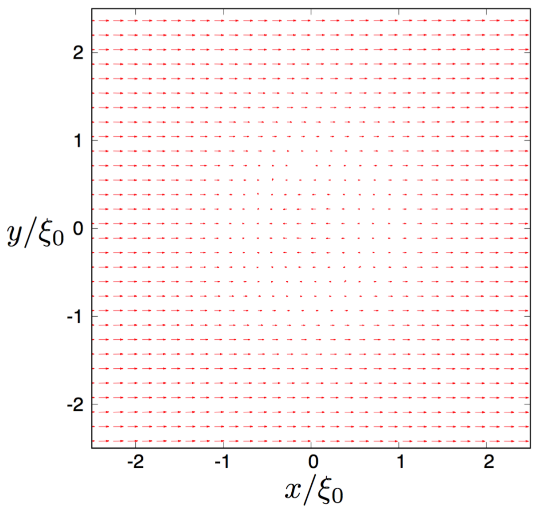

Figure 16: Current density $\delta\boldsymbol{j}^{\mathrm{R}}$ induced by the vortex motion over $-2.5\xi_0 \leq x, y \leq 2.5\xi_0$ at $T/T_c = 0.8$.

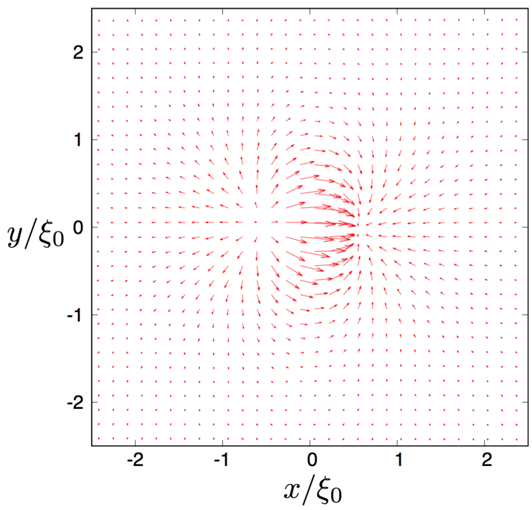

Figure 17: Longitudinal electric field $\delta\boldsymbol{E}_0$ induced by the vortex motion over $-2.5\xi_0 \leq x, y \leq 2.5\xi_0$ at $T/T_c = 0.8$.

Figures 16 and 17 plot the current density $\delta\boldsymbol{j}^{\mathrm{R}}$ and longitudinal electric field $\delta\boldsymbol{E}_0$, respectively, induced by the vortex motion in the core region at $T/T_c = 0.8$. The current density $\delta\boldsymbol{j}^{\mathrm{R}}$, which is obtained from the retarded Green's functions $\delta g_{0\pm}^{\mathrm{R}}$, approaches a

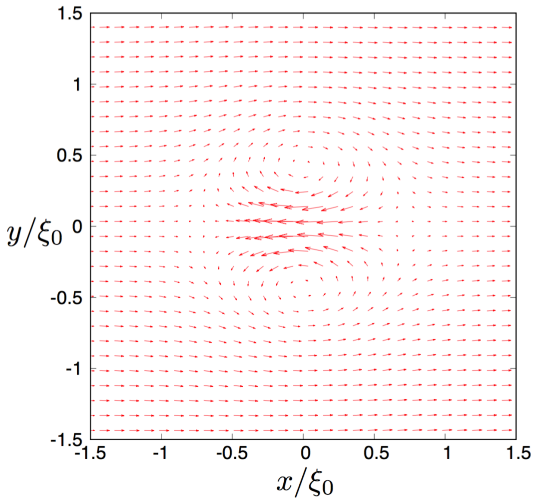

Figure 18: Current density $\delta \boldsymbol{j}^{\mathrm{R}}$ induced by the vortex motion over $-1.5\xi_0 \leq x, y \leq 1.5\xi_0$ at $T/T_{\mathrm{c}} = 0.4$.

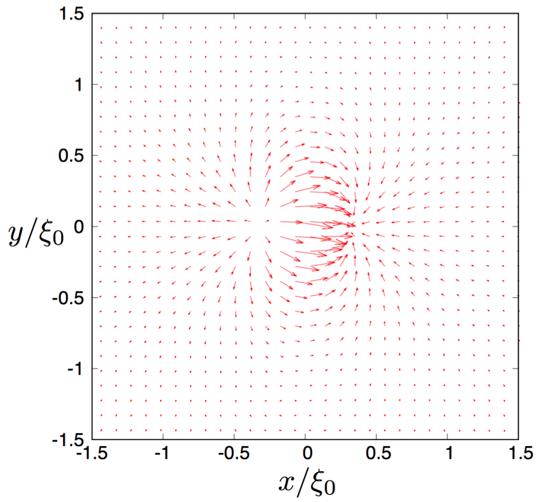

Figure 19: Longitudinal electric field $\delta \boldsymbol{E}_0$ induced by the vortex motion over $-1.5\xi_0 \leq x, y \leq 1.5\xi_0$ at $T/T_{\mathrm{c}} = 0.4$.

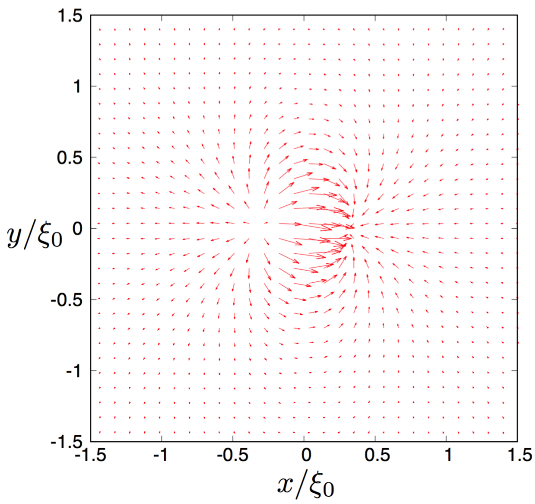

Figure 20: Current density $\delta \boldsymbol{j}^{\mathrm{a}}$ due to the departure from equilibrium over $-1.5\xi_0 \leq x, y \leq 1.5\xi_0$ at $T/T_{\mathrm{c}} = 0.4$.

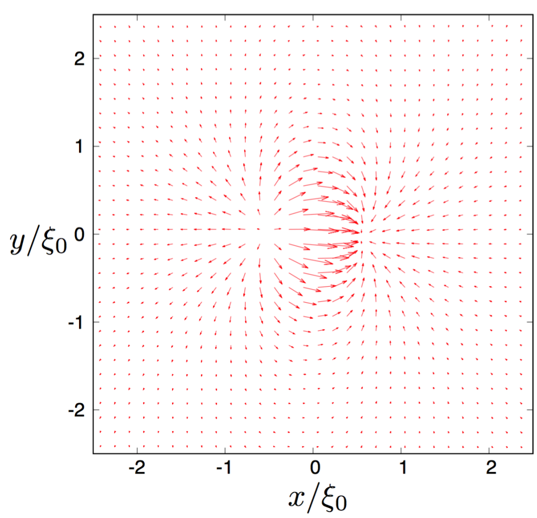

Figure 21: Current density $\delta \boldsymbol{j}^{\mathrm{a}}$ due to the departure from equilibrium over $-2.5\xi_0 \leq x, y \leq 2.5\xi_0$ at $T/T_{\mathrm{c}} = 0.8$.

constant vector $\delta \boldsymbol{j}^{\mathrm{R}}(r \to \infty) \equiv \delta \boldsymbol{j}_{\mathrm{tr}}$ far away from the vortex core.

Figures 18 and 19 plot the current density $\delta \boldsymbol{j}^{\mathrm{R}}$ and electric field $\delta \boldsymbol{E}$, respectively, at a lower temperature $T/T_{\mathrm{c}} = 0.4$. We observe that the current density $\delta \boldsymbol{j}^{\mathrm{R}}$ here exhibits more complicated spatial profile compared with the case of $T/T_{\mathrm{c}} = 0.8$. Also, the region where $\delta \boldsymbol{j}^{\mathrm{R}}$ and $\delta \boldsymbol{E}_0$ vary significantly are smaller here compared with the case of $T/T_{\mathrm{c}} = 0.8$.

Figures 20 and 21 plot the current density $\delta \boldsymbol{j}^{\mathrm{a}}$ due to the departure from equilibrium at $T/T_{\mathrm{c}} = 0.4$ and 0.8, respectively. The current density $\delta \boldsymbol{j}^{\mathrm{a}}$ induced by the electric field, as is the case with the normal state, is obtained from the anomalous Green's functions $\delta g^{\mathrm{a}}_{0\pm}$, and vanishes outside the vortex core. We see that there exists the ohmic resistivity by the moving isolated vortex. The ohmic resistivity at $T/T_{\mathrm{c}} = 0.4$ may be smaller compared with the case of $T/T_{\mathrm{c}} = 0.8$.

4.1.2 Hall electric field and Hall angle

Figures 22 and 23 plot the Hall electric field $\delta \boldsymbol{E}_1$ induced by the vortex motion in the core region at $T/T_{\mathrm{c}} = 0.4$ and 0.8, respectively. Thus, we have confirmed the existence of the flux-flow Hall effect theoretically. Figure 24 plots the Hall angle as a function of temperature. The Hall angle is defined as $\tan\theta_{\mathrm{H}} \equiv \langle E_{\mathrm{H}} \rangle / \langle E_{\mathrm{O}} \rangle$, where $\langle \cdots \rangle$ denotes the spatial average, $\langle E_{\mathrm{O}} \rangle = \langle \delta E_{0x} \rangle$ is the spatial average of the longitudinal electric field and $\langle E_{\mathrm{H}} \rangle = \langle \delta E_{1y} \rangle$ is the spatial average of the Hall electric field. We observe an increase of the Hall angle as the temperature is lowered from $T = T_c$. This may be because the supercurrent, which contributes only to the Hall electric field, becomes more and more dominant compared with the dissipative normal current around the core.

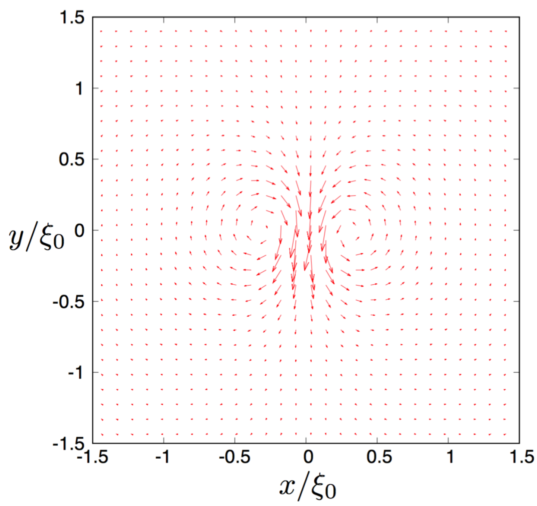

Figure 22: Hall electric field $\delta \boldsymbol{E}_1$ induced by the vortex motion over $-1.5\xi_0 \le x, y \le 1.5\xi_0$ at $T/T_{\mathrm{c}} = 0.4$.

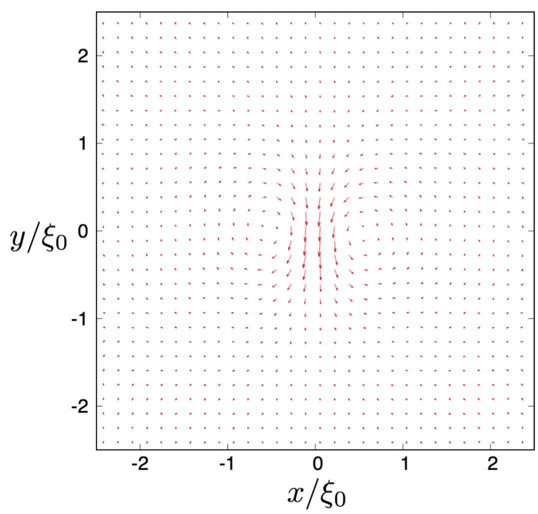

Figure 23: Hall electric field $\delta \boldsymbol{E}_1$ induced by the vortex motion over $-2.5\xi_0 \le x, y \le 2.5\xi_0$ at $T/T_{\mathrm{c}} = 0.8$.

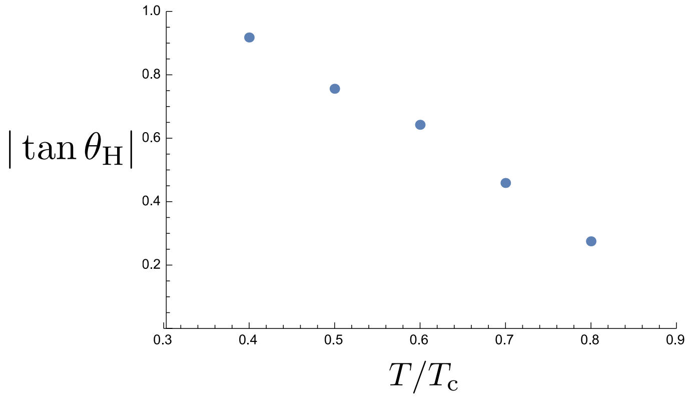

Figure 24: Absolute value of Hall angle $|\tan\theta_{\mathrm{H}}| \equiv |\langle E_{\mathrm{H}}\rangle|/|\langle E_{\mathrm{O}}\rangle|$ over $0.4T_{\mathrm{c}} \leq T \leq 0.8T_{\mathrm{c}}$.

5 Summary and Conclusion

We have studied the vortex-core charging and flux-flow Hall effect based on the augmented quasiclassical equations of superconductivity with the Lorentz force. Previous microscopic calculations of the vortex-core charging have been performed based on the Bogoliubov–de Gennes equations, with which we encountered difficulties in describing superconductors with complicated gap and/or Fermi surface anisotropies. Suitable to this end may be the augmented quasiclassical equations. We have developed a microscopic approach that the vortex-core charge can be estimated easily based on the augmented quasiclassical equations of superconductivity with the Lorentz force in the Matsubara formalism. We also have derived an analytic expression for the vortex-core charge of an isolated vortex in extreme type-II materials given in terms of the London penetration depth and the equilibrium Hall coefficient. Using it, we have observed that the vortex-core charge of an isolated vortex in d-wave superconductors with the isotropic Fermi surface changes sign even as a function of temperature due to the variation in the excitation curvature under the growing energy gap. We hope that our study will trigger further experimental interests on the vortex-core charging.

We have also developed a new approach to calculate the liner responses in the flux-flow state by transforming the energy variable of the augmented quasiclassical equations in the Keldysh formalism into the Matsubara energy on the imaginary axis. Using it, we confirmed that there exists the ohmic and Hall resistivity caused by the moving isolated vortex in an s-wave superconductor. We have found that the region of the spatial variation for both the current density and electric field induced by the vortex motion becomes smaller as temperature is decreased. Our results on the flux-flow Hall effect is consistent with the numerical calculation performed by Arahata and Kato. We have also calculated the Hall angle in flux-flow state as a function of temperature and have observed the increase of the Hall angle as the temperature is lowered from $T = T_c$.

To the best of our knowledge, few studies have been performed on microscopic calculations of the flux-flow Hall effect. This thesis has developed a method to study this complicated topic fully microscopically in a tractable manner.

Appendix

A Derivation of Augmented Quasiclassical Equations in Matsubara formalism

Using the static gauge $\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\Phi(\boldsymbol{r})$ and $\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r})$, we can also derive the augmented quasiclassical equations of superconductivity with the Lorentz force in the equilibrium Matsubara formalism.

A.1 Matsubara Green's functions and Gor'kov equations

We consider conduction electrons in the grand canonical ensemble described by Hamiltonian $\hat{\mathcal{H}}$ with static electromagnetic fields, which are expressed here in terms of the static scalar potential $\Phi(\boldsymbol{r})$ and vector potential $\boldsymbol{A}(\boldsymbol{r})$ as $\boldsymbol{E}(\boldsymbol{r}) = -\boldsymbol{\nabla}\Phi(\boldsymbol{r})$ and $\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{r})$. Let us distinguish the creation and annihilation operators for electrons with integer subscripts $i = 1, 2$ as $\hat{\psi}_1(\xi) \equiv \hat{\psi}(\xi)$ and $\hat{\psi}_2(\xi) \equiv \hat{\psi}^\dagger(\xi)$ [7], where $\xi \equiv (\boldsymbol{r}, \alpha)$ with $\boldsymbol{r}$ and α denoting the space and spin coordinates, respectively. Next, we introduce their Heisenberg representations by $\hat{\psi}_i(1) \equiv e^{\tau_1 \hat{\mathcal{H}}} \hat{\psi}_i(\xi_1) e^{-\tau_1 \hat{\mathcal{H}}}$, where the argument 1 in the round brackets denotes $1 \equiv (\xi_1, \tau_1)$, and the variable τ_1 lies in $0 \leq \tau_1 \leq 1/k_{\mathrm{B}}T$ with k_{B} and T denoting the Boltzmann constant and temperature, respectively. Using them, we introduce the Matsubara Green's function:

$$G_{ij}(1,2) \equiv -\langle T_\tau \hat{\psi}_i(1) \hat{\psi}_{3-j}(2) \rangle, \tag{A.1}$$

where T_τ is the "time"-ordering operator and $\langle \cdots \rangle$ denotes the grand-canonical average [49]. It can be expanded as

$$G_{ij}(1,2) = k_{\mathrm{B}}T \sum_{n=-\infty}^{\infty} G_{ij}(\xi_1, \xi_2; \varepsilon_n) e^{-i\varepsilon_n(\tau_1 - \tau_2)}, \tag{A.2}$$

where $\varepsilon_n = (2n+1)\pi k_{\mathrm{B}}T$ is the fermion Matsubara energy ($n = 0, \pm 1, \ldots$). Separating the spin variable $\alpha = \uparrow, \downarrow$ from $\xi = (\boldsymbol{r}, \alpha)$, we introduce a new notation for each G_{ij} as

$$G_{11}(\xi_1, \xi_2; \varepsilon_n) = G_{\alpha_1,\alpha_2}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n), \tag{A.3a}$$

$$G_{12}(\xi_1, \xi_2; \varepsilon_n) = F_{\alpha_1,\alpha_2}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n), \tag{A.3b}$$

$$G_{21}(\xi_1, \xi_2; \varepsilon_n) = -\bar{F}_{\alpha_1,\alpha_2}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n), \tag{A.3c}$$

$$G_{22}(\xi_1, \xi_2; \varepsilon_n) = -\bar{G}_{\alpha_1,\alpha_2}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n). \tag{A.3d}$$

Subsequently, we express the spin degrees of freedom as the 2×2 matrix

$$\underline{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) \equiv \begin{bmatrix} G_{\uparrow\uparrow}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) & G_{\uparrow\downarrow}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) \\ G_{\downarrow\uparrow}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) & G_{\downarrow\downarrow}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) \end{bmatrix}. \tag{A.4}$$

In matrix notation, $\underline{G}$ and $\underline{F}$ satisfy the following symmetry relations: [7]

$$\underline{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) = \underline{G}^\dagger(\boldsymbol{r}_2, \boldsymbol{r}_1; -\varepsilon_n) = \underline{\bar{G}}^{\mathrm{T}}(\boldsymbol{r}_2, \boldsymbol{r}_1; -\varepsilon_n), \tag{A.5a}$$

$$\underline{F}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) = -\underline{\bar{F}}^\dagger(\boldsymbol{r}_2, \boldsymbol{r}_1; -\varepsilon_n) = -\underline{F}^{\mathrm{T}}(\boldsymbol{r}_2, \boldsymbol{r}_1; -\varepsilon_n), \tag{A.5b}$$

where † and $^{\mathrm{T}}$ denote the Hermitian conjugate and transpose, respectively. It follows from these symmetry relations that $\underline{\bar{G}}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) = \underline{G}^*(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n)$ and $\underline{\bar{F}}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) = \underline{F}^*(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n)$ hold, where superscript * denotes the complex conjugate. Using $\underline{G}$ and $\underline{F}$, we define a 4×4 Nambu matrix by

$$\hat{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) \equiv \begin{bmatrix} \underline{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) & \underline{F}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) \\ -\underline{F}^*(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) & -\underline{G}^*(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) \end{bmatrix}. \tag{A.6}$$

In the mean-field approximation, they satisfy the Gor'kov equations: [50, 7]

$$\begin{aligned} &\begin{bmatrix} (i\varepsilon_n - \hat{\mathcal{K}}_1)\underline{\sigma}_0 & \underline{0} \\ \underline{0} & (i\varepsilon_n + \hat{\mathcal{K}}_1^*)\underline{\sigma}_0 \end{bmatrix} \hat{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) \\ &\quad - \int d^3 r_3 \hat{\mathcal{U}}_{\mathrm{BdG}}(\boldsymbol{r}_1, \boldsymbol{r}_3) \hat{G}(\boldsymbol{r}_3, \boldsymbol{r}_2; \varepsilon_n) = \hat{\delta}(\boldsymbol{r}_1 - \boldsymbol{r}_2), \end{aligned} \tag{A.7}$$

where $\underline{\sigma}_0$ and $\underline{0}$ denote the 2×2 unit and zero matrices, respectively. Operator $\hat{\mathcal{K}}_1$ is defined by

$$\hat{\mathcal{K}}_1 \equiv \frac{1}{2m} \left[-i\hbar \frac{\partial}{\partial \boldsymbol{r}_1} - e\boldsymbol{A}(\boldsymbol{r}_1) \right]^2 + e\Phi(\boldsymbol{r}_1) - \mu, \tag{A.8}$$

where m is the electron mass, $e < 0$ is the electron charge, and μ is the chemical potential. Matrix $\hat{\mathcal{U}}_{\mathrm{BdG}}(\boldsymbol{r}_1, \boldsymbol{r}_3)$ denotes

$$\hat{\mathcal{U}}_{\mathrm{BdG}}(\boldsymbol{r}_1, \boldsymbol{r}_2) \equiv \begin{bmatrix} \underline{\mathcal{U}}_{\mathrm{HF}}(\boldsymbol{r}_1, \boldsymbol{r}_2) & \underline{\Delta}(\boldsymbol{r}_1, \boldsymbol{r}_2) \\ -\underline{\Delta}^*(\boldsymbol{r}_1, \boldsymbol{r}_2) & -\underline{\mathcal{U}}_{\mathrm{HF}}^*(\boldsymbol{r}_1, \boldsymbol{r}_2) \end{bmatrix}, \tag{A.9}$$

where $\underline{\mathcal{U}}_{\mathrm{HF}}$ is the Hartree-Fock potential and $\underline{\Delta}$ is the pair potential [7]. Finally, matrix $\hat{\delta}$ on the right-hand side of Eq. (A.7) is defined by

$$\hat{\delta}(\boldsymbol{r}_1 - \boldsymbol{r}_2) \equiv \begin{bmatrix} \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\underline{\sigma}_0 & \underline{0} \\ \underline{0} & \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)\underline{\sigma}_0 \end{bmatrix}. \tag{A.10}$$

A.2 Gauge invariance

Equation (A.7) has an important property called *gauge invariance* [7].

We introduce the gauge transformation in terms of a continuously differentiable function $\chi(\boldsymbol{r})$ by

$$\begin{cases} \boldsymbol{A}(\boldsymbol{r}_1) = \boldsymbol{A}'(\boldsymbol{r}_1) + \dfrac{\partial \chi(\boldsymbol{r}_1)}{\partial \boldsymbol{r}_1} \\ \hat{\psi}_1(1) = \hat{\psi}_1'(1) \mathrm{e}^{ie\chi(\boldsymbol{r}_1)/\hbar} \\ \hat{\psi}_2(1) = \hat{\psi}_2'(1) \mathrm{e}^{-ie\chi(\boldsymbol{r}_1)/\hbar} \end{cases}, \tag{A.11a}$$

where a prime $'$ distinguishes f' from f as different functions. The corresponding variations of the Green's function (A.6) and potential (A.9) are given by

$$\hat{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) = \hat{\Theta}(\boldsymbol{r}_1)\hat{G}'(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n)\hat{\Theta}^*(\boldsymbol{r}_2), \tag{A.12a}$$

$$\hat{\mathcal{U}}_{\rm BdG}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \hat{\Theta}(\boldsymbol{r}_1)\hat{\mathcal{U}}'_{\rm BdG}(\boldsymbol{r}_1, \boldsymbol{r}_2)\hat{\Theta}^*(\boldsymbol{r}_2), \tag{A.12b}$$

where matrix $\hat{\Theta}$ is defined by

$$\hat{\Theta}(\boldsymbol{r}_1) \equiv \begin{bmatrix} \underline{\sigma}_0 e^{ie\chi(\boldsymbol{r}_1)/\hbar} & \underline{0} \\ \underline{0} & \underline{\sigma}_0 e^{-ie\chi(\boldsymbol{r}_1)/\hbar} \end{bmatrix}. \tag{A.13}$$

Moreover, using $[\mp i\hbar\partial/\partial\boldsymbol{r}_1 - e\boldsymbol{A}(\boldsymbol{r}_1)]^2 e^{\pm ie\chi(\boldsymbol{r}_1)/\hbar} = e^{\pm ie\chi(\boldsymbol{r}_1)/\hbar}[\mp i\hbar\partial/\partial\boldsymbol{r}_1 - e\boldsymbol{A}'(\boldsymbol{r}_1)]^2$, the $\hat{\mathcal{K}}_1$ term of Eq. (A.7) is expressible as

$$\begin{bmatrix} -\hat{\mathcal{K}}_1\underline{\sigma}_0 & \underline{0} \\ \underline{0} & \hat{\mathcal{K}}_1^*\underline{\sigma}_0 \end{bmatrix} \hat{\Theta}(\boldsymbol{r}_1) = \hat{\Theta}(\boldsymbol{r}_1) \begin{bmatrix} -\hat{\mathcal{K}}'_1\underline{\sigma}_0 & \underline{0} \\ \underline{0} & \hat{\mathcal{K}}'^*_1\underline{\sigma}_0 \end{bmatrix}. \tag{A.14}$$

Let us substitute Eqs. (A.12a) and (A.12b) into Eq. (A.7), then use Eq. (A.14), and multiply the resulting equation by $\hat{\Theta}^*(\boldsymbol{r}_1)$ and $\hat{\Theta}(\boldsymbol{r}_2)$ from left and right. We then realize that the resulting equation in terms of $\boldsymbol{A}'(\boldsymbol{r}_1)$, $\hat{G}'(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n)$ and $\hat{\mathcal{U}}'_{\rm BdG}(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is identical in form to Eq. (A.7). This is *gauge invariance*, implying that there is an arbitrariness in the choice of vector potential.

A.3 Gauge-covariant Wigner transform

The original *Wigner transform* [51] may be defined, for example, in terms of the Nambu matrix (A.6) as follows: Let us introduce the "center-of-mass" and "relative" coordinates as

$$\boldsymbol{r}_{12} = \frac{\boldsymbol{r}_1 + \boldsymbol{r}_2}{2}, \qquad \bar{\boldsymbol{r}}_{12} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2. \tag{A.15}$$

The Wigner transform is defined as the Fourier transform with respect to the relative coordinates,

$$\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}_{12}) \equiv \int d^3\bar{r}_{12} e^{-i\boldsymbol{p}\cdot\bar{\boldsymbol{r}}_{12}/\hbar}\hat{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n), \tag{A.16}$$

where $\hat{G}$'s on both sides are different functions distinguished by their arguments. However, the original Wigner transform breaks the gauge invariance with respect to the center-of-mass coordinate when applied to the Green's functions of charged systems. To remove this drawback, Stratonovich introduced a modified Wigner transform that may be called the *gauge-invariant Wigner transform* [52]. However, the method is valid only for normal systems with $\underline{G}_{12} = \underline{G}_{21} = 0$. Here, we apply an extended version for describing superconductors [7, 8].

First, we introduce the line integral

$$I(\boldsymbol{r}_1, \boldsymbol{r}_2) \equiv \frac{e}{\hbar}\int_{\boldsymbol{r}_2}^{\boldsymbol{r}_1} \boldsymbol{A}(\boldsymbol{s}) \cdot d\boldsymbol{s}, \tag{A.17}$$

where $\boldsymbol{s}$ denotes a straight-line path from $\boldsymbol{r}_2$ to $\boldsymbol{r}_1$. Next, we define matrix $\hat{\Gamma}$ by

$$\hat{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2) \equiv \begin{bmatrix} \underline{\sigma}_0 e^{iI(\boldsymbol{r}_1,\boldsymbol{r}_2)} & \underline{0} \\ \underline{0} & \underline{\sigma}_0 e^{-iI(\boldsymbol{r}_1,\boldsymbol{r}_2)} \end{bmatrix}. \tag{A.18}$$

Now, the *gauge-covariant Wigner transform* for the Green's functions Eq. (A.6) is defined by

$$\begin{aligned} &\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}_{12}) \\ &\equiv \int d^3\bar{r}_{12} e^{-i\boldsymbol{p}\cdot\bar{\boldsymbol{r}}_{12}/\hbar} \hat{\Gamma}(\boldsymbol{r}_{12}, \boldsymbol{r}_1) \hat{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) \hat{\Gamma}(\boldsymbol{r}_2, \boldsymbol{r}_{12}) \\ &\equiv \begin{bmatrix} \underline{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}_{12}) & \underline{F}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}_{12}) \\ -\underline{F}^*(\varepsilon_n, -\boldsymbol{p}, \boldsymbol{r}_{12}) & -\underline{G}^*(\varepsilon_n, -\boldsymbol{p}, \boldsymbol{r}_{12}) \end{bmatrix}, \end{aligned} \tag{A.19a}$$

the inverse of which is given by

$$\hat{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) = \hat{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_{12}) \int \frac{d^3p}{(2\pi\hbar)^3} e^{i\boldsymbol{p}\cdot\bar{\boldsymbol{r}}_{12}/\hbar} \hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}_{12}) \hat{\Gamma}(\boldsymbol{r}_{12}, \boldsymbol{r}_2). \tag{A.19b}$$

It can be shown easily that $\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}_{12})$ changes under the gauge transformation in Eq. (A.12a) to

$$\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}_{12}) = \hat{\Theta}(\boldsymbol{r}_{12}) \hat{G}'(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}_{12}) \hat{\Theta}^*(\boldsymbol{r}_{12}). \tag{A.20}$$

Thus, only the center-of-mass coordinate is relevant to the variation of $\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}_{12})$ under the gauge transformation.

Similarly, we transform the mean-field potential (A.9)

$$\begin{aligned} &\hat{\mathcal{U}}_{\rm BdG}(\boldsymbol{p}, \boldsymbol{r}_{12}) \\ &\equiv \int d^3\bar{r}_{12} e^{-i\boldsymbol{p}\cdot\bar{\boldsymbol{r}}_{12}/\hbar} \hat{\Gamma}(\boldsymbol{r}_{12}, \boldsymbol{r}_1) \hat{\mathcal{U}}_{\rm BdG}(\boldsymbol{r}_1, \boldsymbol{r}_2) \hat{\Gamma}(\boldsymbol{r}_2, \boldsymbol{r}_{12}) \\ &\equiv \begin{bmatrix} \underline{\mathcal{U}}_{\rm HF}(\boldsymbol{p}, \boldsymbol{r}_{12}) & \underline{\Delta}(\boldsymbol{p}, \boldsymbol{r}_{12}) \\ -\underline{\Delta}^*(-\boldsymbol{p}, \boldsymbol{r}_{12}) & -\underline{\mathcal{U}}_{\rm HF}^*(-\boldsymbol{p}, \boldsymbol{r}_{12}) \end{bmatrix}, \end{aligned} \tag{A.21a}$$

whose inverse reads

$$\hat{\mathcal{U}}_{\rm BdG}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \hat{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_{12}) \int \frac{d^3p}{(2\pi\hbar)^3} e^{i\boldsymbol{p}\cdot\bar{\boldsymbol{r}}_{12}/\hbar} \hat{\mathcal{U}}_{\rm BdG}(\boldsymbol{p}, \boldsymbol{r}_{12}) \hat{\Gamma}(\boldsymbol{r}_{12}, \boldsymbol{r}_2). \tag{A.21b}$$

Note that potentials $\underline{\mathcal{U}}_{\rm HF}(\boldsymbol{p}, \boldsymbol{r}_{12})$ and $\underline{\Delta}(\boldsymbol{p}, \boldsymbol{r}_{12})$ satisfy the following relations: $\underline{\mathcal{U}}_{\rm HF}(\boldsymbol{p}, \boldsymbol{r}_{12}) = \underline{\mathcal{U}}_{\rm HF}^\dagger(\boldsymbol{p}, \boldsymbol{r}_{12})$ and $\underline{\Delta}(\boldsymbol{p}, \boldsymbol{r}_{12}) = -\underline{\Delta}^{\rm T}(-\boldsymbol{p}, \boldsymbol{r}_{12})$.

A.4 Derivation of augmented quasiclassical equations

With these preliminaries, we derive the augmented quasiclassical equations in the Matsubara formalism following the procedure in Ref. [8] for the Keldysh formalism.

Let us introduce the functions

$$\mathcal{E}_1(u) \equiv \int_0^1 d\eta e^{\eta u} = \frac{e^u - 1}{u}, \tag{A.22a}$$

$$\mathcal{E}_2(u) \equiv \int_0^1 d\eta \int_0^\eta d\zeta e^{\zeta u} = \frac{e^u - 1 - u}{u^2}. \tag{A.22b}$$

The line integral in Eq. (A.17) and its partial derivatives are expressible in terms of these functions as

$$I(\boldsymbol{r}_1, \boldsymbol{r}_{12}) = \frac{e}{\hbar}\mathcal{E}_1\left(\frac{\bar{\boldsymbol{r}}_{12}}{2}\cdot\frac{\partial}{\partial \boldsymbol{r}_{12}}\right)\frac{\bar{\boldsymbol{r}}_{12}}{2}\cdot\boldsymbol{A}(\boldsymbol{r}_{12}), \tag{A.23}$$

$$\begin{aligned}\frac{\partial}{\partial \boldsymbol{r}_1}I(\boldsymbol{r}_1, \boldsymbol{r}_{12}) &= \frac{e}{\hbar}\boldsymbol{A}(\boldsymbol{r}_1) - \frac{e}{2\hbar}\boldsymbol{A}(\boldsymbol{r}_{12}) \\ &\quad - \frac{e}{4\hbar}\left[2\mathcal{E}_1\left(\frac{\bar{\boldsymbol{r}}_{12}}{2}\cdot\frac{\partial}{\partial \boldsymbol{r}_{12}}\right) - \mathcal{E}_2\left(\frac{\bar{\boldsymbol{r}}_{12}}{2}\cdot\frac{\partial}{\partial \boldsymbol{r}_{12}}\right)\right]\left[\boldsymbol{B}(\boldsymbol{r}_{12})\times\bar{\boldsymbol{r}}_{12}\right],\end{aligned} \tag{A.24a}$$

$$\frac{\partial}{\partial \boldsymbol{r}_1}I(\boldsymbol{r}_{12}, \boldsymbol{r}_2) = \frac{e}{2\hbar}\boldsymbol{A}(\boldsymbol{r}_{12}) - \frac{e}{4\hbar}\mathcal{E}_2\left(-\frac{\bar{\boldsymbol{r}}_{12}}{2}\cdot\frac{\partial}{\partial \boldsymbol{r}_{12}}\right)\left[\boldsymbol{B}(\boldsymbol{r}_{12})\times\bar{\boldsymbol{r}}_{12}\right]. \tag{A.24b}$$

Now, we focus on the kinetic-energy terms in Eq. (A.7) given by

$$\begin{bmatrix}\hat{\mathcal{K}}_1\underline{\sigma}_0 & \underline{0} \\ \underline{0} & -\hat{\mathcal{K}}_1^*\underline{\sigma}_0\end{bmatrix}\hat{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) = \begin{bmatrix}\hat{\mathcal{K}}_1\underline{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) & \hat{\mathcal{K}}_1\underline{F}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) \\ \hat{\mathcal{K}}_1^*\underline{F}^*(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) & \hat{\mathcal{K}}_1^*\underline{G}^*(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n)\end{bmatrix}. \tag{A.25}$$

Substituting Eq.(A.19b) and using Eq. (A.24), we can transform each submatrix on the right-hand side as

$$\begin{aligned}\hat{\mathcal{K}}_1\underline{G}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) &\approx e^{-iI(\boldsymbol{r}_{12},\boldsymbol{r}_1)}e^{iI(\boldsymbol{r}_{12},\boldsymbol{r}_2)}\int\frac{d^3p}{(2\pi\hbar)^3}e^{i\boldsymbol{p}\cdot\bar{\boldsymbol{r}}_{12}/\hbar} \\ &\quad\times\left\{\xi_p + e\Phi(\boldsymbol{r}_{12}) - \frac{i\hbar}{2}\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}_{12}} - \frac{i\hbar}{2}e\boldsymbol{E}(\boldsymbol{r}_{12})\cdot\boldsymbol{\partial}_{\boldsymbol{p}}\right. \\ &\quad\left.-\frac{i\hbar}{2}e\boldsymbol{v}\cdot\left[\boldsymbol{B}(\boldsymbol{r}_{12})\times\boldsymbol{\partial}_{\boldsymbol{p}}\right]\right\}\underline{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}_{12}),\end{aligned} \tag{A.26a}$$

$$\begin{aligned}\hat{\mathcal{K}}_1\underline{F}(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) &\approx e^{-iI(\boldsymbol{r}_{12},\boldsymbol{r}_1)}e^{-iI(\boldsymbol{r}_{12},\boldsymbol{r}_2)}\int\frac{d^3p}{(2\pi\hbar)^3}e^{i\boldsymbol{p}\cdot\bar{\boldsymbol{r}}_{12}/\hbar} \\ &\quad\times\left\{\xi_p + e\Phi(\boldsymbol{r}_{12}) - \frac{i\hbar}{2}\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}_{12}} - e\boldsymbol{v}\cdot\boldsymbol{A}(\boldsymbol{r}_{12})\right. \\ &\quad\left.-\frac{i\hbar}{2}e\boldsymbol{E}(\boldsymbol{r}_{12})\cdot\boldsymbol{\partial}_{\boldsymbol{p}} - \frac{i\hbar}{4}e\boldsymbol{v}\cdot\left[\boldsymbol{B}(\boldsymbol{r}_{12})\times\boldsymbol{\partial}_{\boldsymbol{p}}\right]\right\}\underline{F}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}_{12}),\end{aligned} \tag{A.26b}$$

$$\begin{aligned}\hat{\mathcal{K}}_1^* \underline{F}^*(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) \approx\ & e^{iI(\boldsymbol{r}_{12},\boldsymbol{r}_1)} e^{iI(\boldsymbol{r}_{12},\boldsymbol{r}_2)} \int \frac{d^3p}{(2\pi\hbar)^3} e^{i\boldsymbol{p}\cdot\bar{\boldsymbol{r}}_{12}/\hbar} \\ & \times \left\{ \xi_p + e\Phi(\boldsymbol{r}_{12}) - \frac{i\hbar}{2}\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}_{12}} + e\boldsymbol{v}\cdot\boldsymbol{A}(\boldsymbol{r}_{12}) \right. \\ & \left. - \frac{i\hbar}{2} e\boldsymbol{E}(\boldsymbol{r}_{12})\cdot\boldsymbol{\partial}_{\boldsymbol{p}} + \frac{i\hbar}{4} e\boldsymbol{v}\cdot[\boldsymbol{B}(\boldsymbol{r}_{12})\times\boldsymbol{\partial}_{\boldsymbol{p}}] \right\} \\ & \times \underline{F}^*(\varepsilon_n, -\boldsymbol{p}, \boldsymbol{r}_{12}), \end{aligned} \tag{A.26c}$$

$$\begin{aligned}\hat{\mathcal{K}}_1^* \underline{G}^*(\boldsymbol{r}_1, \boldsymbol{r}_2; \varepsilon_n) \approx\ & e^{iI(\boldsymbol{r}_{12},\boldsymbol{r}_1)} e^{-iI(\boldsymbol{r}_{12},\boldsymbol{r}_2)} \int \frac{d^3p}{(2\pi\hbar)^3} e^{i\boldsymbol{p}\cdot\bar{\boldsymbol{r}}_{12}/\hbar} \\ & \times \left\{ \xi_p + e\Phi(\boldsymbol{r}_{12}) - \frac{i\hbar}{2}\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}_{12}} - \frac{i\hbar}{2} e\boldsymbol{E}(\boldsymbol{r}_{12})\cdot\boldsymbol{\partial}_{\boldsymbol{p}} \right. \\ & \left. + \frac{i\hbar}{2} e\boldsymbol{v}\cdot[\boldsymbol{B}(\boldsymbol{r}_{12})\times\boldsymbol{\partial}_{\boldsymbol{p}}] \right\} \underline{G}^*(\varepsilon_n, -\boldsymbol{p}, \boldsymbol{r}_{12}), \end{aligned} \tag{A.26d}$$

where $\xi_p \equiv p^2/2m - \mu$, and $\boldsymbol{\partial}$ is defined by

$$\boldsymbol{\partial} \equiv \begin{cases} \boldsymbol{\nabla} & \text{on } \underline{G} \text{ or } \underline{G}^* \\ \boldsymbol{\nabla} - i\dfrac{2e\boldsymbol{A}}{\hbar} & \text{on } \underline{F} \\ \boldsymbol{\nabla} + i\dfrac{2e\boldsymbol{A}}{\hbar} & \text{on } \underline{F}^* \end{cases} . \tag{A.27a}$$

The following approximations have been adopted in deriving Eq. (A.26): (i) We have neglected spatial derivatives of both $\boldsymbol{E}$ and $\boldsymbol{B}$, which amounts to setting $\mathcal{E}_1 \to 1$ and $\mathcal{E}_2 \to 1/2$. (ii) We also have neglected terms second-order in $\boldsymbol{\partial}_{\boldsymbol{r}_{12}}$, $\boldsymbol{E}$, and $\boldsymbol{B}$. (iii) We have expanded Φ around $\boldsymbol{r}_{12}$ up to the first order in $\bar{\boldsymbol{r}}_{12}$ as $\Phi(\boldsymbol{r}_1) \approx \Phi(\boldsymbol{r}_{12}) - \boldsymbol{E}(\boldsymbol{r}_{12})\cdot\bar{\boldsymbol{r}}_{12}/2$. By these procedures, we obtain the Gor'kov equations in the Wigner representation,

$$\begin{aligned} & i\varepsilon_n \hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) - \left[\xi_p + e\Phi(\boldsymbol{r}) - \frac{i\hbar}{2}\boldsymbol{v}\cdot\boldsymbol{\partial} \right] \hat{\tau}_3 \hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) \\ & + \frac{i\hbar}{2} e\boldsymbol{E}(\boldsymbol{r})\cdot\boldsymbol{\partial}_{\boldsymbol{p}} \hat{\tau}_3 \hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) \\ & + \frac{i\hbar}{8} e\boldsymbol{v}\cdot[\boldsymbol{B}(\boldsymbol{r})\times\boldsymbol{\partial}_{\boldsymbol{p}}] \left[3\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) + \hat{\tau}_3 \hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) \hat{\tau}_3 \right] \\ & - \hat{\mathcal{U}}_{\rm BdG}(\boldsymbol{p}, \boldsymbol{r}) \hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) = \hat{1}, \end{aligned} \tag{A.28}$$

where $\hat{\tau}_3$ is defined by

$$\hat{\tau}_3 \equiv \begin{bmatrix} \underline{\sigma}_0 & \underline{0} \\ \underline{0} & -\underline{\sigma}_0 \end{bmatrix}, \tag{A.29}$$

and $\hat{1}$ denotes the 4×4 unit matrix. We take the Hermitian conjugate of Eq. (A.28), use the symmetries $\hat{\mathcal{U}}^\dagger_{\rm BdG}(\boldsymbol{p}, \boldsymbol{r}) = \hat{\mathcal{U}}_{\rm BdG}(\boldsymbol{p}, \boldsymbol{r})$ and $\hat{G}^\dagger(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) = \hat{G}(-\varepsilon_n, \boldsymbol{p}, \boldsymbol{r})$, and set

$\varepsilon_n \to -\varepsilon_n$ to obtain

$$\begin{aligned}
&i\varepsilon_n \hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) - \left[\xi_p + e\Phi(\boldsymbol{r}) + \frac{i\hbar}{2}\boldsymbol{v}\cdot\boldsymbol{\partial}\right]\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r})\hat{\tau}_3 \\
&\quad - \frac{i\hbar}{2} e\boldsymbol{E}(\boldsymbol{r})\cdot\boldsymbol{\partial}_{\boldsymbol{p}}\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r})\hat{\tau}_3 \\
&\quad - \frac{i\hbar}{8} e\boldsymbol{v}\cdot[\boldsymbol{B}(\boldsymbol{r})\times\boldsymbol{\partial}_{\boldsymbol{p}}]\left[3\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) + \hat{\tau}_3\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r})\hat{\tau}_3\right] \\
&\quad - \hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r})\hat{\mathcal{U}}_{\mathrm{BdG}}(\boldsymbol{p}, \boldsymbol{r}) = \hat{1}.
\end{aligned} \tag{A.30}$$

Equations (A.28) and (A.30) are referred to as the left and right Gor'kov equations, respectively. Now, we operate $\hat{\tau}_3$ from the left and right sides of Eq. (A.30) and subtract the resulting equation from Eq. (A.28). We thereby obtain

$$\begin{aligned}
&\left[i\varepsilon_n\hat{\tau}_3 - \hat{\mathcal{U}}_{\mathrm{BdG}}(\boldsymbol{p}, \boldsymbol{r})\hat{\tau}_3, \hat{\tau}_3\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r})\right] \\
&\quad + i\hbar\boldsymbol{v}\cdot\boldsymbol{\partial}\hat{\tau}_3\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) + i\hbar e\boldsymbol{E}(\boldsymbol{r})\cdot\boldsymbol{\partial}_{\boldsymbol{p}}\hat{\tau}_3\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) \\
&\quad + \frac{i\hbar}{2} e\boldsymbol{v}\cdot[\boldsymbol{B}(\boldsymbol{r})\times\boldsymbol{\partial}_{\boldsymbol{p}}]\left\{\hat{\tau}_3, \hat{\tau}_3\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r})\right\} = \hat{0},
\end{aligned} \tag{A.31}$$

with $[\hat{a}, \hat{b}] \equiv \hat{a}\hat{b} - \hat{b}\hat{a}$ and $\{\hat{a}, \hat{b}\} \equiv \hat{a}\hat{b} + \hat{b}\hat{a}$.

Finally, we perform integration over ξ_p neglecting all the ξ_p dependences except those in $\hat{G}$. To this end, we introduce the quasiclassical Green's functions:

$$\begin{aligned}
\hat{g}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}) &\equiv \mathrm{P}\int_{-\infty}^{\infty}\frac{d\xi_p}{\pi}\hat{\tau}_3 i\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) \\
&\equiv \begin{bmatrix} \underline{g}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}) & -i\underline{f}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}) \\ -i\underline{f}^*(\varepsilon_n, -\boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}) & -\underline{g}^*(\varepsilon_n, -\boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}) \end{bmatrix},
\end{aligned} \tag{A.32}$$

where P denotes the principal value. We also carry out the following procedures to obtain the final equations: (i) Rewrite $\boldsymbol{\partial}_{\mathbf{p}} = \boldsymbol{\partial}_{\mathbf{p}_\parallel} + \boldsymbol{v}_{\mathrm{F}}(\partial/\partial\xi_p)$ with $\mathbf{p}_\parallel$ the component on the energy surface of $\xi_p =$ constant. (ii) Make use of $\boldsymbol{v}_{\mathrm{F}}\times\boldsymbol{\partial}_{\mathbf{p}_\parallel} = \boldsymbol{v}_{\mathrm{F}}\times\boldsymbol{\partial}_{\mathbf{p}}$ and

$$\mathrm{P}\int_{-\infty}^{\infty} d\xi_p \frac{\partial^m}{\partial\xi_p^m}\hat{\tau}_3 i\hat{G}(\varepsilon_n, \boldsymbol{p}, \boldsymbol{r}) = 0, \qquad (m = 1, 2, \cdots).$$

(iii) Neglect the term $\boldsymbol{E}\cdot\boldsymbol{\partial}_{\mathbf{p}_\parallel}$ because it is second-order in the quasiclassical parameter $\delta \equiv \hbar/\langle p_{\mathrm{F}}\rangle_{\mathrm{F}}\xi_0 \ll 1$ [8, 10], where ξ_0 is the coherence length defined in terms of the zero-temperature energy gap $\langle\Delta_0\rangle_{\mathrm{F}}$ at $B = 0$ by $\xi_0 \equiv \hbar\langle v_{\mathrm{F}}\rangle_{\mathrm{F}}/\langle\Delta_0\rangle_{\mathrm{F}}$. We thereby obtain the augmented quasiclassical equations in the Matsubara formalism as

$$\begin{aligned}
&\left[i\varepsilon_n\hat{\tau}_3 - \hat{\mathcal{U}}_{\mathrm{BdG}}(\boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})\hat{\tau}_3, \hat{g}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})\right] + i\hbar\boldsymbol{v}_{\mathrm{F}}\cdot\boldsymbol{\partial}\hat{g}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}) \\
&\quad + \frac{i\hbar}{2} e\boldsymbol{v}_{\mathrm{F}}\cdot[\boldsymbol{B}(\boldsymbol{r})\times\boldsymbol{\partial}_{\boldsymbol{p}_{\mathrm{F}}}]\{\hat{\tau}_3, \hat{g}(\varepsilon_n, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})\} = \hat{0}.
\end{aligned} \tag{A.33}$$

Thus, the electric field is absent from the equations in the Matsubara formalism unlike those in the Keldysh formalism.

Now, we consider the weak-coupling case and include the effects of impurity scatterings in the self-consistent Born approximation by[7] $\hat{\mathcal{U}}_{\text{BdG}}(\boldsymbol{p}_\text{F},\boldsymbol{r})\to\hat{\Delta}(\boldsymbol{p}_\text{F},\boldsymbol{r})+\hat{\sigma}_\text{imp}(\varepsilon_n,\boldsymbol{r})$. The pair potentials $\hat{\Delta}(\boldsymbol{p}_\text{F},\boldsymbol{r})$ and impurity self-energy $\hat{\sigma}_\text{imp}(\varepsilon_n,\boldsymbol{r})$ are given explicitly by

$$\hat{\Delta}(\boldsymbol{p}_\text{F},\boldsymbol{r})\equiv\begin{bmatrix}\underline{0} & \underline{\Delta}(\boldsymbol{p}_\text{F},\boldsymbol{r})\\ -\underline{\Delta}^*(-\boldsymbol{p}_\text{F},\boldsymbol{r}) & \underline{0}\end{bmatrix}, \tag{A.34a}$$

$$\hat{\sigma}_\text{imp}(\varepsilon_n,\boldsymbol{r})\equiv -i\frac{\hbar}{2\tau}\langle\hat{g}(\varepsilon_n,\boldsymbol{p}_\text{F},\boldsymbol{r})\rangle_\text{F}\hat{\tau}_3, \tag{A.34b}$$

where τ is the relaxation time and $\langle\cdots\rangle_\text{F}$ denotes the Fermi surface average with $\langle 1\rangle_\text{F}=1$. The augmented quasiclassical equations in the Matsubara formalism are then given by

$$\begin{aligned}&\left[i\varepsilon_n\hat{\tau}_3-\hat{\Delta}\hat{\tau}_3-\hat{\sigma}_\text{imp}\hat{\tau}_3,\hat{g}\right]\\ &\quad+i\hbar\boldsymbol{v}_\text{F}\cdot\boldsymbol{\partial}\hat{g}+\frac{i\hbar}{2}e(\boldsymbol{v}_\text{F}\times\boldsymbol{B})\cdot\frac{\partial}{\partial\boldsymbol{p}_\text{F}}\left\{\hat{\tau}_3,\hat{g}\right\}=\hat{0}.\end{aligned} \tag{A.35}$$

Matrices $\hat{g}$ and $\hat{\Delta}$ can be written as [7]

$$\hat{g}\equiv\begin{bmatrix}\underline{g} & -i\underline{f}\\ -i\underline{\bar{f}} & -\underline{\bar{g}}\end{bmatrix},\qquad \hat{\Delta}\equiv\begin{bmatrix}\underline{0} & \underline{\Delta}\\ -\underline{\bar{\Delta}} & \underline{0}\end{bmatrix}, \tag{A.36}$$

where the barred functions are defined generally by $\underline{\bar{g}}(\varepsilon_n,\boldsymbol{p}_\text{F},\boldsymbol{r})\equiv\underline{g}^*(\varepsilon_n,-\boldsymbol{p}_\text{F},\boldsymbol{r})$. It is worth pointing out that the same equations result in the gauge $\boldsymbol{E}(\boldsymbol{r})=-\partial\boldsymbol{A}'(\boldsymbol{r},t)/\partial t$ and $\boldsymbol{B}(\boldsymbol{r})=\boldsymbol{\nabla}\times\boldsymbol{A}'(\boldsymbol{r},t)$ with $\Phi'=0$. The gauge transformation $(\Phi,\boldsymbol{A})\to(0,\boldsymbol{A}')$ is given by

$$\Phi(\boldsymbol{r})=-\frac{\partial\chi(\boldsymbol{r},t)}{\partial t}, \tag{A.37a}$$

$$\boldsymbol{A}(\boldsymbol{r})=\boldsymbol{A}'(\boldsymbol{r},t)+\boldsymbol{\nabla}\chi(\boldsymbol{r},t), \tag{A.37b}$$

$$\underline{g}(\varepsilon_n,\boldsymbol{p}_\text{F},\boldsymbol{r})=\underline{g}'(\varepsilon_n,\boldsymbol{p}_\text{F},\boldsymbol{r}), \tag{A.37c}$$

$$\underline{f}(\varepsilon_n,\boldsymbol{p}_\text{F},\boldsymbol{r})=\underline{f}'(\varepsilon_n,\boldsymbol{p}_\text{F},\boldsymbol{r})e^{2ie\chi(\boldsymbol{r},t)/\hbar}, \tag{A.37d}$$

where the continuously differentiable function $\chi(\boldsymbol{r},t)$ is fixed by

$$\boldsymbol{\nabla}\chi(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r})t,\qquad\frac{\partial\boldsymbol{\nabla}\chi(\boldsymbol{r},t)}{\partial t}=-\frac{\partial\boldsymbol{A}'(\boldsymbol{r},t)}{\partial t}. \tag{A.38}$$

A.5 Analytic Continuation in Terms of Frequency

Next, we consider the augmented quasiclassical equations in the Keldysh formalism and study their connection with Eq. (A.35). It is convenient when describing equilibrium states in the Keldysh formalism to set $\Phi'\to 0$ and express static electromagnetic fields in terms of only the vector potential $\boldsymbol{A}'$ with linear time dependence as $\boldsymbol{E}(\boldsymbol{r})=-\partial\boldsymbol{A}'(\boldsymbol{r},t)/\partial t$ and $\boldsymbol{B}(\boldsymbol{r})=\boldsymbol{\nabla}\times\boldsymbol{A}'(\boldsymbol{r},t)$. The rationale for this is that the scalar potential Φ' in the Keldysh formalism always appears in the covariant form $i\hbar\partial/\partial t-2e\Phi'$,[8] which in the present gauge can be set equal to zero naturally for static situations. Thus, we derive the augmented quasiclassical equations in the Keldysh formalism in the static case using the following line integral:

$$I(\vec{r}_1, \vec{r}_2) \equiv -\frac{e}{\hbar} \int_{\vec{r}_2}^{\vec{r}_1} \vec{A}(\vec{s}) \cdot d\vec{s}, \tag{A.39}$$

where $I(\boldsymbol{r}_1, \boldsymbol{r}_2)$ defined by Eq. (A.17) and $I(\vec{r}_1, \vec{r}_2)$ defined by Eq. (A.39) are different functions distinguished by their arguments, $\vec{r}_1 \equiv (t_1, \boldsymbol{r}_1)$ is the four-vector, and $d\vec{s}$ is taken along the straight line, and $\vec{A}(\boldsymbol{r}, t)$ is given by

$$\vec{A}(\boldsymbol{r}, t) \equiv \left(-\frac{\partial \chi(\boldsymbol{r}, t)}{\partial t}, -\boldsymbol{A}'(\boldsymbol{r}, t) - \boldsymbol{\nabla} \chi(\boldsymbol{r}, t) \right), \tag{A.40}$$

where $\chi(\boldsymbol{r}, t)$ is also fixed as Eq. (A.38). The gauge-covariant Wigner transform for the retarded Green's functions is now given by

$$\begin{aligned}
&\hat{G}^{\mathrm{R}}(\varepsilon, \boldsymbol{p}, \boldsymbol{r}_{12}) \\
&\equiv \int d^3\bar{r}_{12} d\bar{t}_{12} e^{-i(\boldsymbol{p}\cdot\bar{\boldsymbol{r}}_{12} - \varepsilon \bar{t}_{12})/\hbar} \hat{\Gamma}(\vec{r}_{12}, \vec{r}_1) \hat{G}^{\mathrm{R}}(\boldsymbol{r}_1, \boldsymbol{r}_2; \bar{t}_{12}) \hat{\Gamma}(\vec{r}_2, \vec{r}_{12}) \\
&\equiv \begin{bmatrix} \underline{G}^{\mathrm{R}}(\varepsilon, \boldsymbol{p}, \boldsymbol{r}_{12}) & \underline{F}^{\mathrm{R}}(\varepsilon, \boldsymbol{p}, \boldsymbol{r}_{12}) \\ -\underline{F}^{\mathrm{R}*}(-\varepsilon, -\boldsymbol{p}, \boldsymbol{r}_{12}) & -\underline{G}^{\mathrm{R}*}(-\varepsilon, -\boldsymbol{p}, \boldsymbol{r}_{12}) \end{bmatrix},
\end{aligned} \tag{A.41}$$

where $t_{12} \equiv (t_1 + t_2)/2$, $\bar{t}_{12} \equiv t_1 - t_2$, and matrix $\hat{\Gamma}$ is defined by

$$\hat{\Gamma}(\vec{r}_1, \vec{r}_2) \equiv \begin{bmatrix} \underline{\sigma}_0 e^{iI(\vec{r}_1, \vec{r}_2)} & \underline{0} \\ \underline{0} & \underline{\sigma}_0 e^{-iI(\vec{r}_1, \vec{r}_2)} \end{bmatrix}. \tag{A.42}$$

The corresponding augmented quasiclassical equations for the retarded submatrix $\hat{g}^{\mathrm{R}} = \hat{g}^{\mathrm{R}}(\varepsilon, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})$ are given by [8, 10]

$$\begin{aligned}
&\left[\varepsilon \hat{\tau}_3 - \hat{\Delta} \hat{\tau}_3 - \hat{\sigma}^{\mathrm{R}}_{\mathrm{imp}} \hat{\tau}_3, \hat{g}^{\mathrm{R}}\right] + i\hbar \boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{\partial} \hat{g}^{\mathrm{R}} \\
&\quad + \frac{i\hbar}{2} \left[e\boldsymbol{v}_{\mathrm{F}} \cdot \boldsymbol{E} \frac{\partial}{\partial \varepsilon} + e(\boldsymbol{v}_{\mathrm{F}} \times \boldsymbol{B}) \cdot \frac{\partial}{\partial \boldsymbol{p}_{\mathrm{F}}} \right] \left\{ \hat{\tau}_3, \hat{g}^{\mathrm{R}} \right\} = \hat{0},
\end{aligned} \tag{A.43a}$$

$$\hat{\sigma}^{\mathrm{R}}_{\mathrm{imp}} \equiv -\frac{i\hbar}{2\tau} \langle \hat{g}^{\mathrm{R}} \rangle_{\mathrm{F}} \hat{\tau}_3. \tag{A.43b}$$

The quasiclassical Green's function $\hat{g}^{\mathrm{R}}$ is expressible as [8, 10]

$$\hat{g}^{\mathrm{R}} = \begin{bmatrix} \underline{g}^{\mathrm{R}} & -i\underline{f}^{\mathrm{R}} \\ -i\underline{\bar{f}}^{\mathrm{R}} & -\underline{\bar{g}}^{\mathrm{R}} \end{bmatrix}, \tag{A.44}$$

where each barred 2×2 submatrix is connected generally to its unbarred equivalent as $\underline{\bar{g}}^{\mathrm{R}}(\varepsilon, \boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r}) = \underline{g}^{\mathrm{R}*}(-\varepsilon, -\boldsymbol{p}_{\mathrm{F}}, \boldsymbol{r})$. Thus, Eq. (A.43) manifestly contains an electric-field term, which is absent in Eq. (A.35), however. The issue here is how to perform the analytic continuation between $\hat{g}'$ and $\hat{g}^{\mathrm{R}}$ obeying Eqs. (A.35) and (A.43) with different forms. Alternatively, one may depend solely on Eq. (A.43) and put $\varepsilon \to i\varepsilon_n$ directly;

however, this procedure also has a difficulty in how to perform differentiation with respect to ε_n, which has discrete values.

To find the procedure, we extract the (1,1) and (1,2) submatrix elements from Eq. (A.43). They can be written explicitly as

$$\begin{aligned}
&\hbar \boldsymbol{v}_{\rm F} \cdot \boldsymbol{\nabla} \underline{g}^{\rm R} + \hbar e \boldsymbol{v}_{\rm F} \cdot \boldsymbol{E} \frac{\partial \underline{g}^{\rm R}}{\partial \varepsilon} + \hbar e (\boldsymbol{v}_{\rm F} \times \boldsymbol{B}) \cdot \frac{\partial \underline{g}^{\rm R}}{\partial \mathbf{p}_{\rm F}} \\
&\quad - \underline{\Delta} \underline{\bar{f}}^{\rm R} + \underline{f}^{\rm R} \underline{\bar{\Delta}} + \frac{\hbar}{2\tau} \left(\underline{f}^{\rm R} \langle \underline{\bar{f}}^{\rm R} \rangle_{\rm F} - \langle \underline{f}^{\rm R} \rangle_{\rm F} \underline{\bar{f}}^{\rm R} \right) = \underline{0},
\end{aligned} \tag{A.45a}$$

$$\begin{aligned}
&- 2i\varepsilon \underline{f}^{\rm R} + \hbar \boldsymbol{v}_{\rm F} \cdot \left(\boldsymbol{\nabla} - i \frac{2e\boldsymbol{A}'}{\hbar} \right) \underline{f}^{\rm R} - \underline{\Delta} \underline{\bar{g}}^{\rm R} - \underline{g}^{\rm R} \underline{\Delta} \\
&\quad + \frac{\hbar}{2\tau} \left(\langle \underline{g}^{\rm R} \rangle_{\rm F} \underline{f}^{\rm R} - \langle \underline{f}^{\rm R} \rangle_{\rm F} \underline{\bar{g}}^{\rm R} - \underline{g}^{\rm R} \langle \underline{f}^{\rm R} \rangle_{\rm F} + \underline{f}^{\rm R} \langle \underline{\bar{g}}^{\rm R} \rangle_{\rm F} \right) = \underline{0}.
\end{aligned} \tag{A.45b}$$

We then write the gradient term in Eq. (A.45a) together with the electric-field term as

$$\boldsymbol{\nabla} \underline{g}^{\rm R} + e \boldsymbol{E} \frac{\partial \underline{g}^{\rm R}}{\partial \varepsilon} \equiv \boldsymbol{\nabla} \underline{\tilde{g}}^{\rm R}, \tag{A.46}$$

and eliminate $\underline{g}^{\rm R}$ in the two equations in favor of $\underline{\tilde{g}}^{\rm R}$. We then use (i) the smallness of the Lorentz term by $\delta \ll 1$ [10]. (ii) $\underline{g}^{\rm R} \propto \underline{\sigma}_0$ for the leading order and (iii) $\bar{g}^{\rm R} - g^{\rm R} = O(\delta)$, to neglect terms of $O(\delta^2)$. The procedure yields

$$\begin{aligned}
&\hbar \boldsymbol{v}_{\rm F} \cdot \boldsymbol{\nabla} \underline{\tilde{g}}^{\rm R} + \hbar e (\boldsymbol{v}_{\rm F} \times \boldsymbol{B}) \cdot \frac{\partial \underline{\tilde{g}}^{\rm R}}{\partial \mathbf{p}_{\rm F}} \\
&\quad - \underline{\Delta} \underline{\bar{f}}^{\rm R} + \underline{f}^{\rm R} \underline{\bar{\Delta}} + \frac{\hbar}{2\tau} \left(\underline{f}^{\rm R} \langle \underline{\bar{f}}^{\rm R} \rangle_{\rm F} - \langle \underline{f}^{\rm R} \rangle_{\rm F} \underline{\bar{f}}^{\rm R} \right) = \underline{0},
\end{aligned} \tag{A.47a}$$

$$\begin{aligned}
&- 2i\varepsilon \underline{f}^{\rm R} + \hbar \boldsymbol{v}_{\rm F} \cdot \left(\boldsymbol{\nabla} - i \frac{2e\boldsymbol{A}'}{\hbar} \right) \underline{f}^{\rm R} - \underline{\Delta} \underline{\bar{\tilde{g}}}^{\rm R} - \underline{\tilde{g}}^{\rm R} \underline{\Delta} \\
&\quad + \frac{\hbar}{2\tau} \left(\langle \underline{\tilde{g}}^{\rm R} \rangle_{\rm F} \underline{f}^{\rm R} - \langle \underline{f}^{\rm R} \rangle_{\rm F} \underline{\bar{\tilde{g}}}^{\rm R} - \underline{\tilde{g}}^{\rm R} \langle \underline{f}^{\rm R} \rangle_{\rm F} + \underline{f}^{\rm R} \langle \underline{\bar{\tilde{g}}}^{\rm R} \rangle_{\rm F} \right) = \underline{0}.
\end{aligned} \tag{A.47b}$$

These equations are identical in form with those for $(\underline{g}', \underline{f}')$ from Eq. (A.35) transformed by Eq. (A.37), as can be seen easily. This implies that we may perform the analytic continuation in terms of $\varepsilon_n > 0$ using

$$\begin{cases} \underline{g}'(\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r}) = \underline{\tilde{g}}^{\rm R}(i\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r}) \\ \underline{f}'(\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r}) = \underline{f}^{\rm R}(i\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r}) \end{cases}. \tag{A.48}$$

Accordingly, the expression for the charge density in the Matsubara formalism needs to be modified. To see this, we start from the expression in the Keldysh formalism [45, 53]:

$$\rho = -\frac{eN(0)}{4} \int_{-\infty}^{\infty} {\rm Tr} \langle \underline{g}^{\rm K} \rangle_{\rm F} \, d\varepsilon.$$

Here, $N(0)$ is the normal density of states per spin and unit volume at the Fermi energy, Tr denotes the trace in spin space, and $\underline{g}^{\rm K} = (\underline{g}^{\rm R} - \underline{g}^{\rm A}) \tanh(\varepsilon / 2k_{\rm B}T)$ in equilibrium with

$\underline{g}^{\rm A} \equiv -\underline{\sigma}_3 \underline{g}^{\rm R\dagger} \underline{\sigma}_3$, where $\underline{\sigma}_3$ denotes the third Pauli matrix. Let us apply the operater $\boldsymbol{\nabla}$ to this equation, substitute Eq. (A.46), and use $\mathrm{Tr}\, \underline{g}^{\rm K} \to \pm 4$ for $\varepsilon \to \pm\infty$ to perform integration with respect to ε for the electric-field term. This leads to

$$\begin{aligned} \boldsymbol{\nabla}\rho = &-\frac{eN(0)}{4}\boldsymbol{\nabla}\int_{-\infty}^{\infty} \mathrm{Tr}\langle\, \underline{\tilde{g}}^{\rm R} - \underline{\tilde{g}}^{\rm A}\rangle_{\rm F} \tanh\frac{\varepsilon}{2k_{\rm B}T}\, d\varepsilon \\ &+ 2e^2 N(0)\boldsymbol{E}. \end{aligned}$$

Deforming the contour of the above integral towards the imaginary axis using the residue theorem, and noting Eq. (A.48), we can express the charge density in terms of $\underline{g}(\varepsilon_n, \boldsymbol{p}_{\rm F}, \boldsymbol{r})$ as

$$\rho = -i\pi k_{\rm B}TeN(0)\sum_{n=-\infty}^{\infty} \mathrm{Tr}\langle \underline{g}\rangle_{\rm F} - 2e^2 N(0)\Phi. \tag{A.49}$$

This expression is the same as that in Refs. [6, 45], and [46]. On the other hand, the formula for the current density has no extra term with $\boldsymbol{E}$ because $\langle \boldsymbol{v}_{\rm F}\rangle_{\rm F} = \boldsymbol{0}$, and so is the equation for the energy gap [7, 45, 53]. This argument is valid even when the impurity self-energy is incorporated. This completes our formulation of the augmented quasiclassical equations in the Matsubara formalism.

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